Design & Analysis of Information Systems

Mathematics in Computer Science.

This year's topic is

Computational Geometry.

Mathematics in computer science

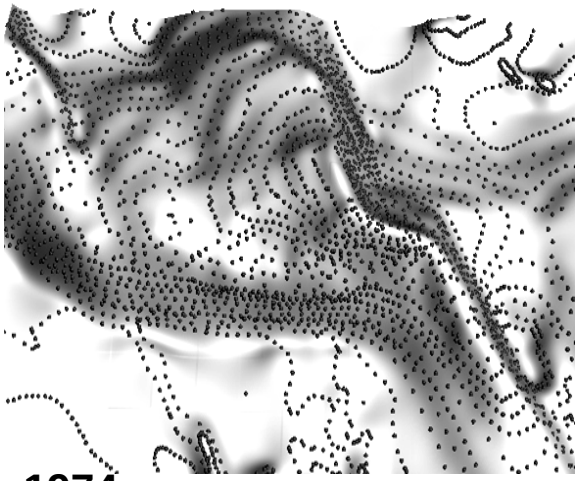
- Mathematics seeks for elegant solutions
  - "Solution" is not just "showing the answer".
  - It is important to describe the solution process.
- Computer science seeks for elegant and efficient algorithms
  - Algorithm: Concrete description of process.
  - Algorithms lead to our modern life.
- Mathematics is vital in algorithm design
  - Solve seemingly-impossible tasks.

Computational Geometry

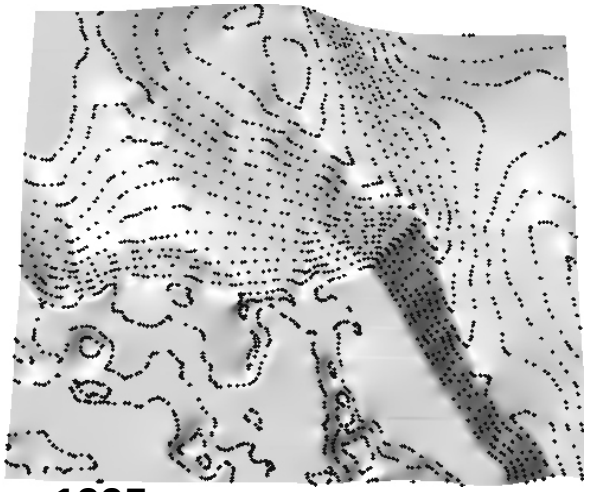
- Design algorithms for geometric problems
  - Many modern applications
    - GIS, Graphics, Geometric Modeling, Robotics, Multidimensional Database, Computer Vision.
  - Fast processing of massive data
    - Giga pixel data = 1000,000,000 data in a single digital picture
- Elegant geometry for algorithm design
  - Discrete geometry, etc.
  - Exciting intellectual puzzles (知的パズル)

Diverse elevation point data: density, distribution, accuracy

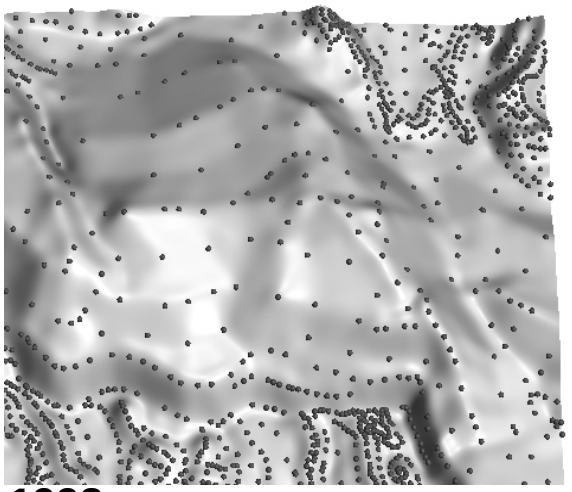
Photogrammetry 0.76m v. accuracy (5ft contours)



1974



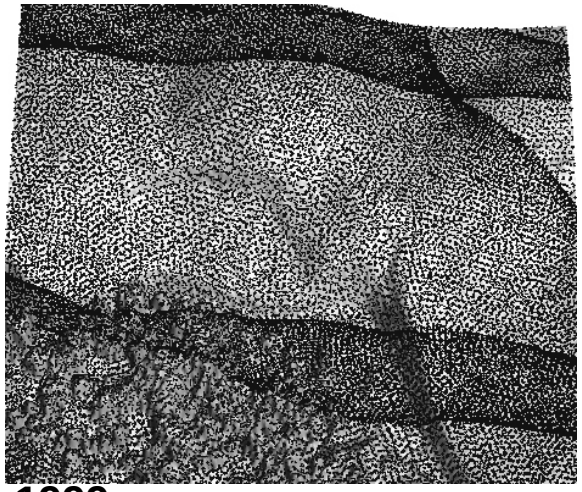
1995



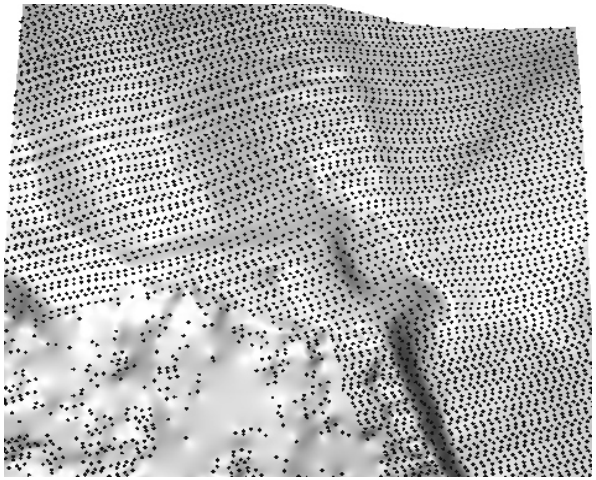
1998

Lidar 0.15m v. accuracy; altitude 700m and 2300m

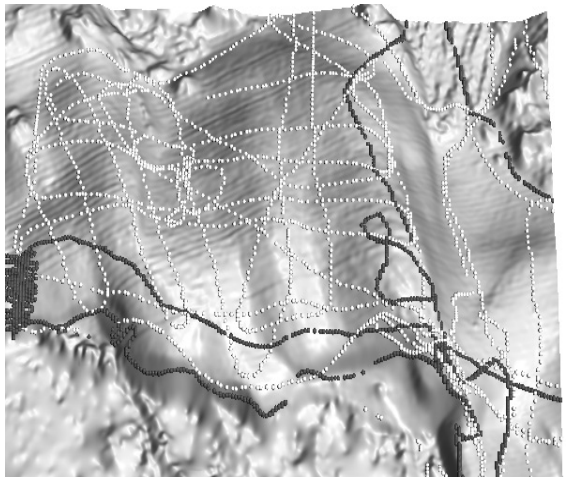
RTK-GPS 0.10m v. accuracy



1999



2001



2004

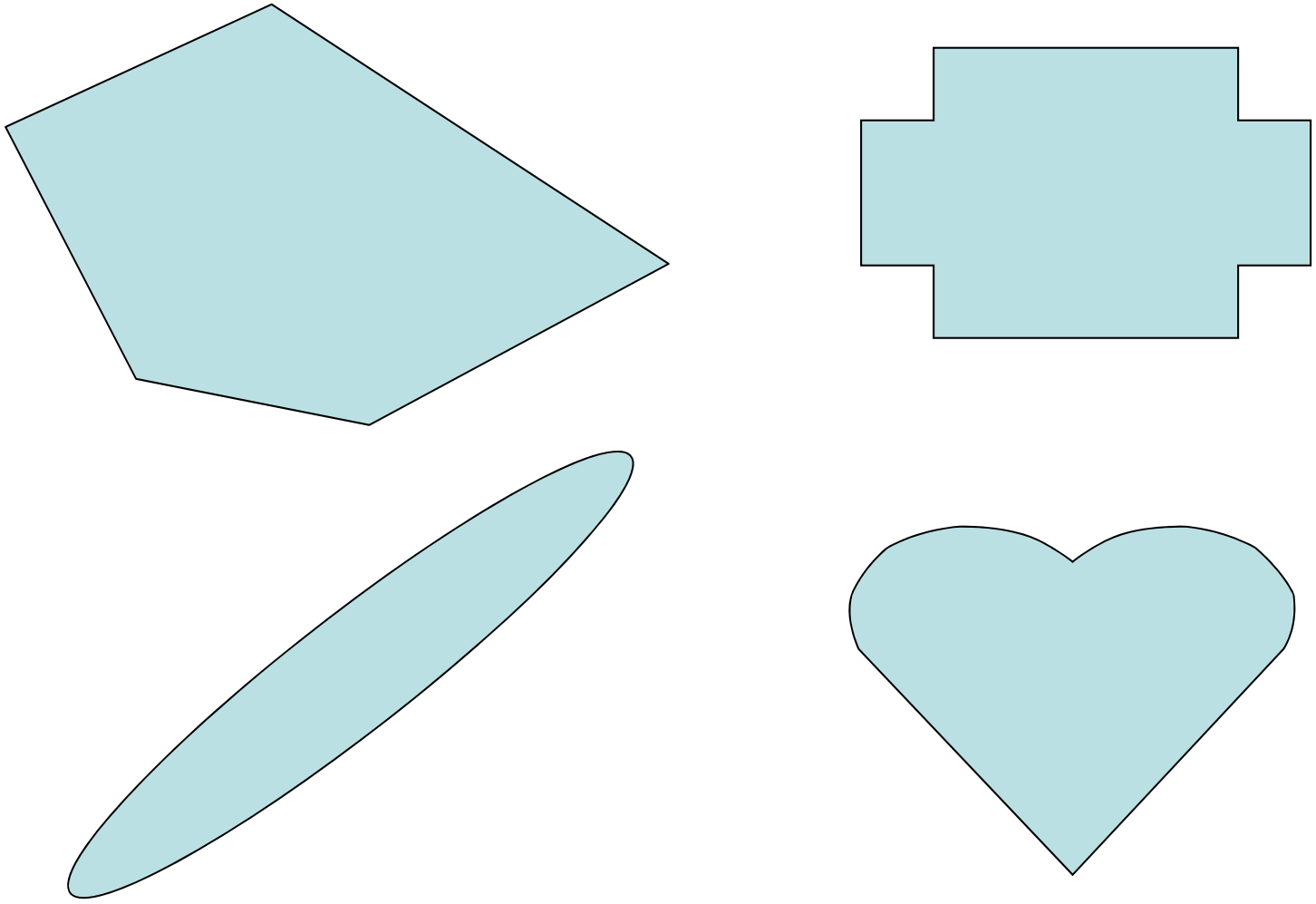
N

100 meters

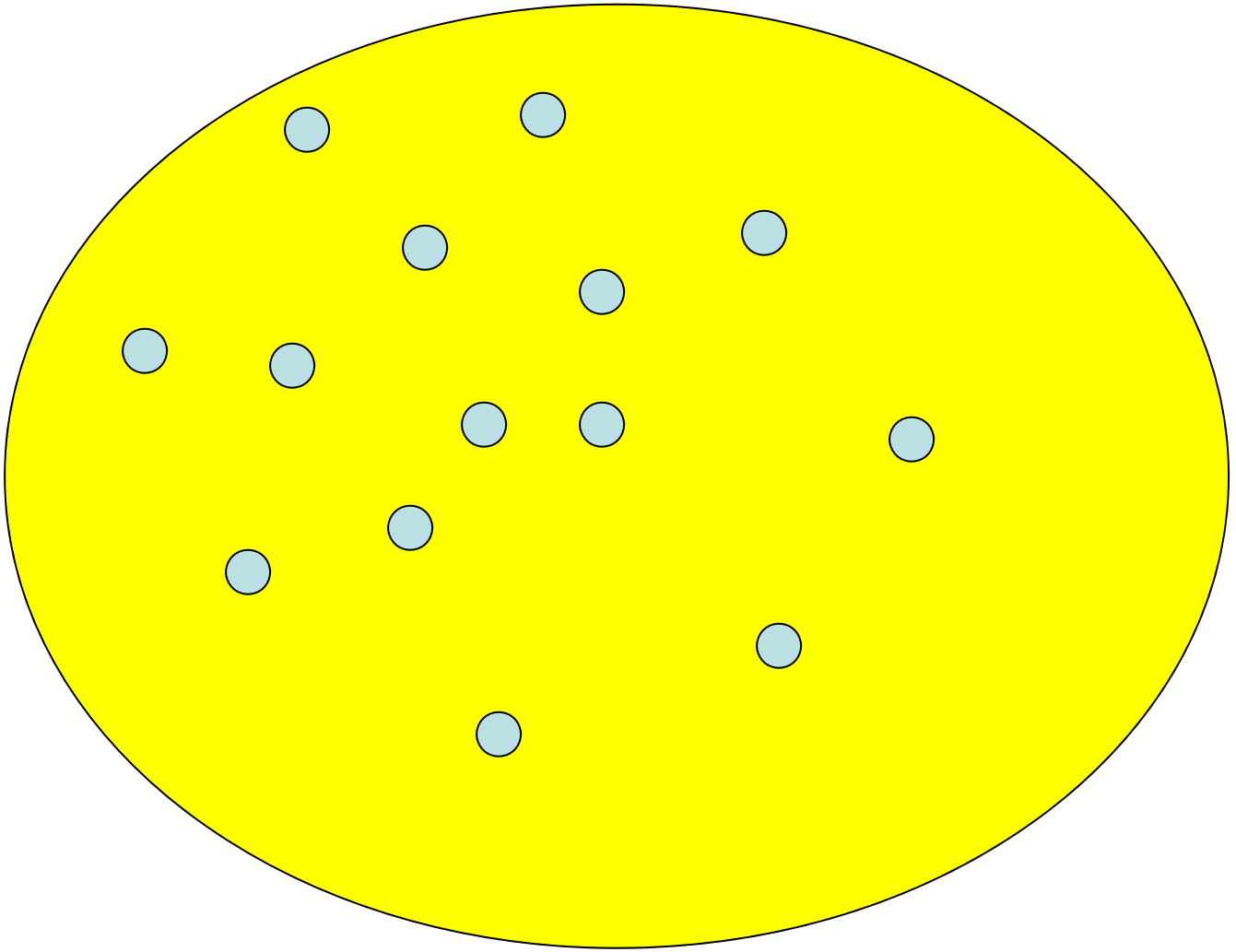
Example of geometric computation

- Convex hull computation（凸包の計算）
  - A showcase of algorithmic techniques
- Given a set S of n points in a plane, compute its convex hull
  - Convex set: A set X such that for any two points p and q in X, the segment pq is in X
  - Convex hull CH(S) of S : Minimum convex set containing S
    - Question: Is convex hull well-defined (i.e., always uniquely exists) ?

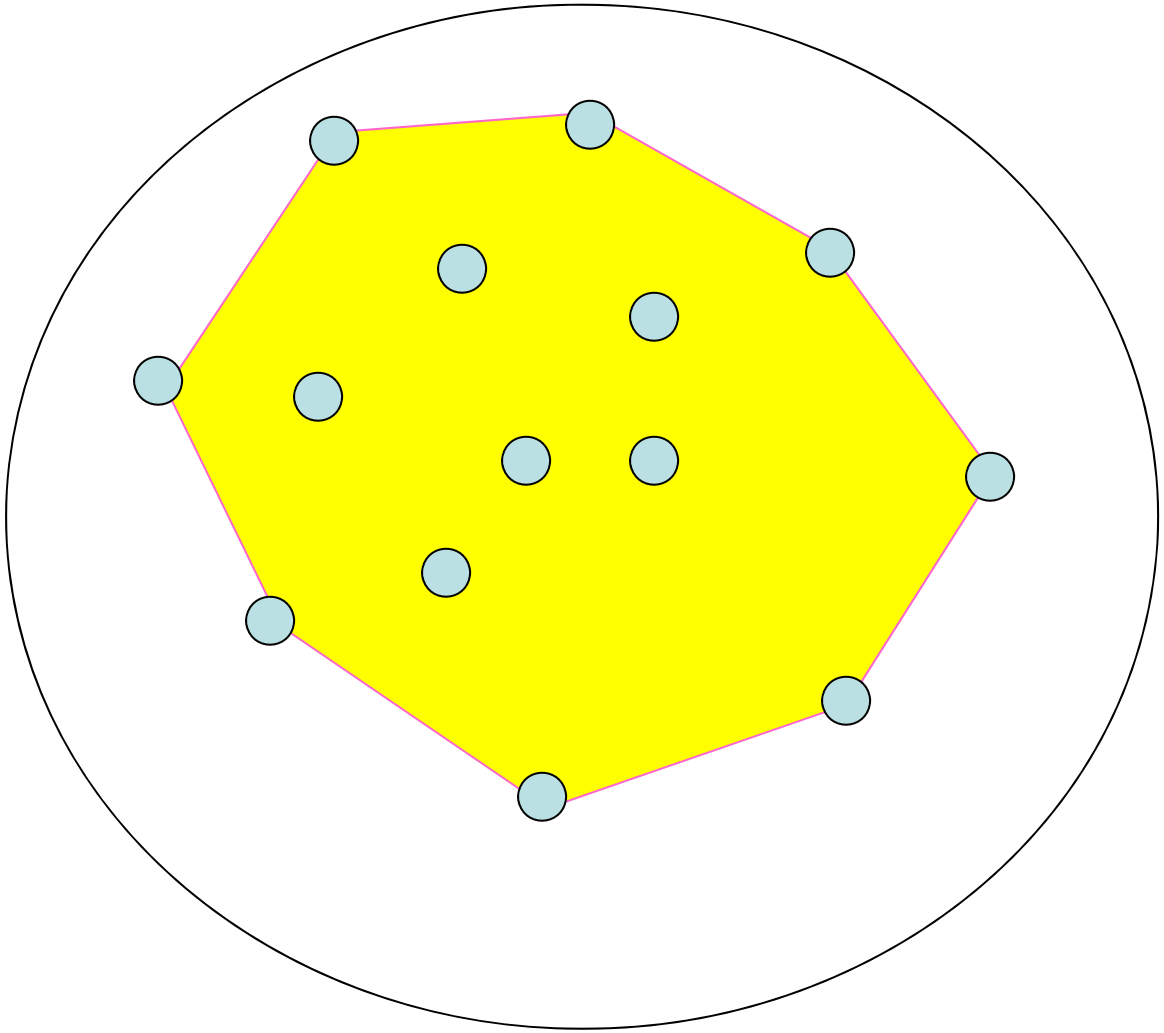
Convex Sets ?



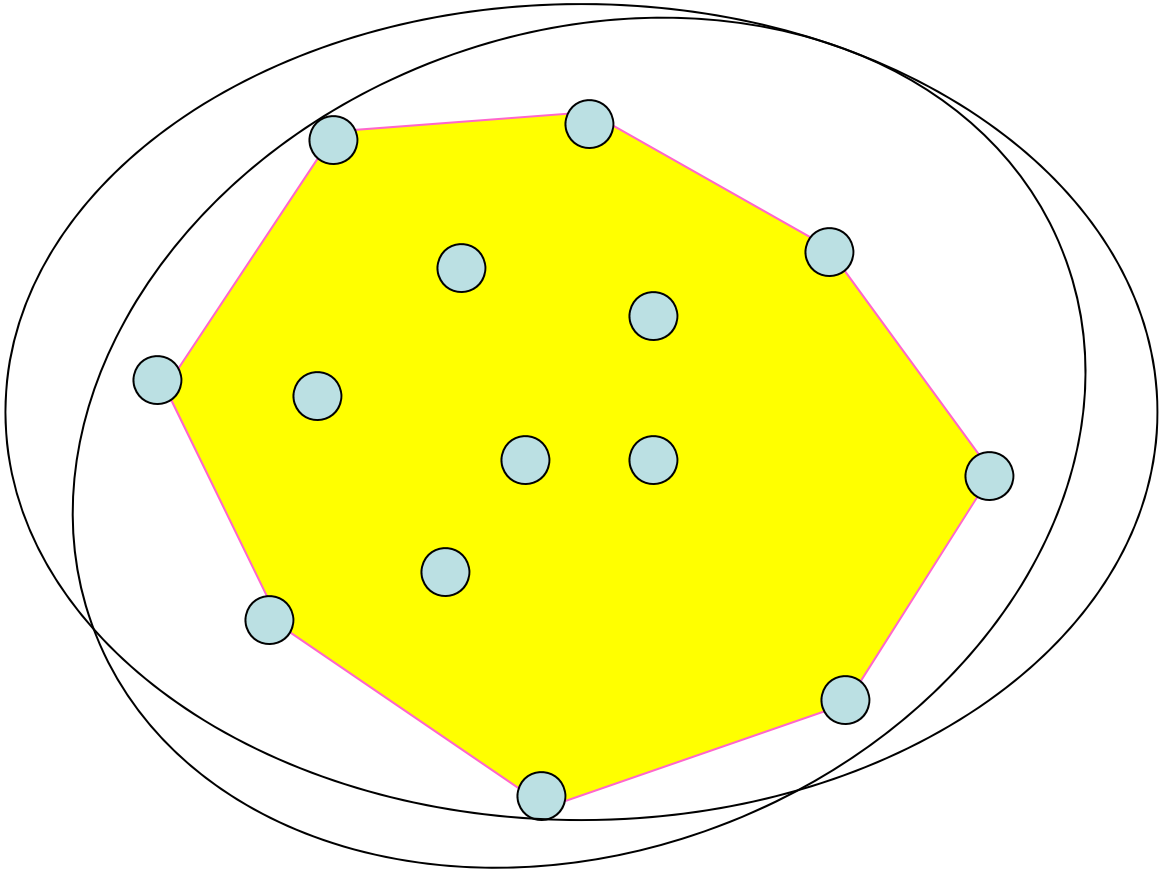
Convex set containing S



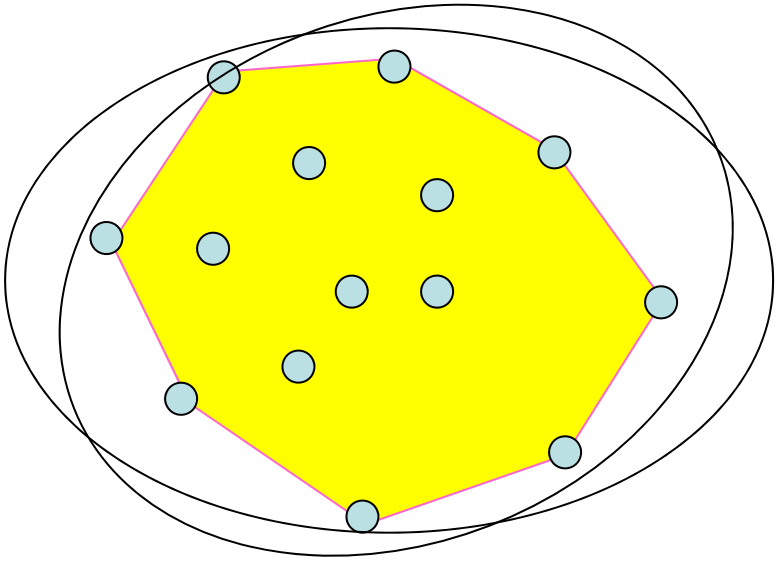
Convex hull of S



Every convex set containing S must also contain CH(S)



Convex hull exists



$$CH(S) = \bigcap_{X : convex,\ X \supset S} X$$

- Nice mathematical representation.
- But it is hard to use in computation.

Convex Hull Computation

- Given a set S of n points, compute CH(S)
- Can you design an algorithm?

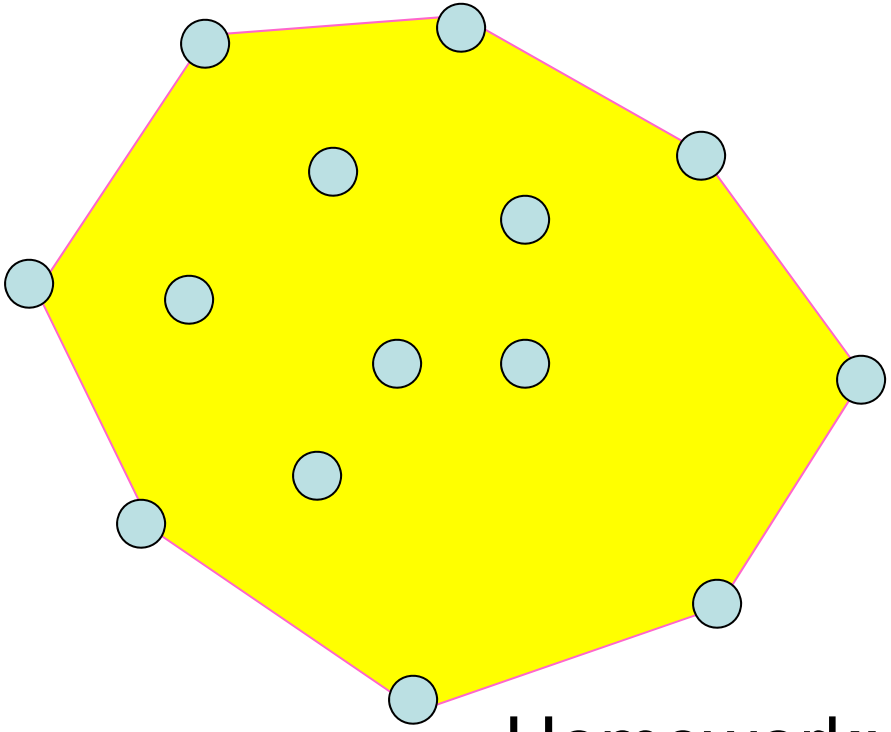
$$CH(S) = \bigcap_{X:convex, X \supset S} X$$

Very nuisance formula. The right hand side is intersection of infinite number of convex sets

We should transform this “cold” formula into more “friendly” one.

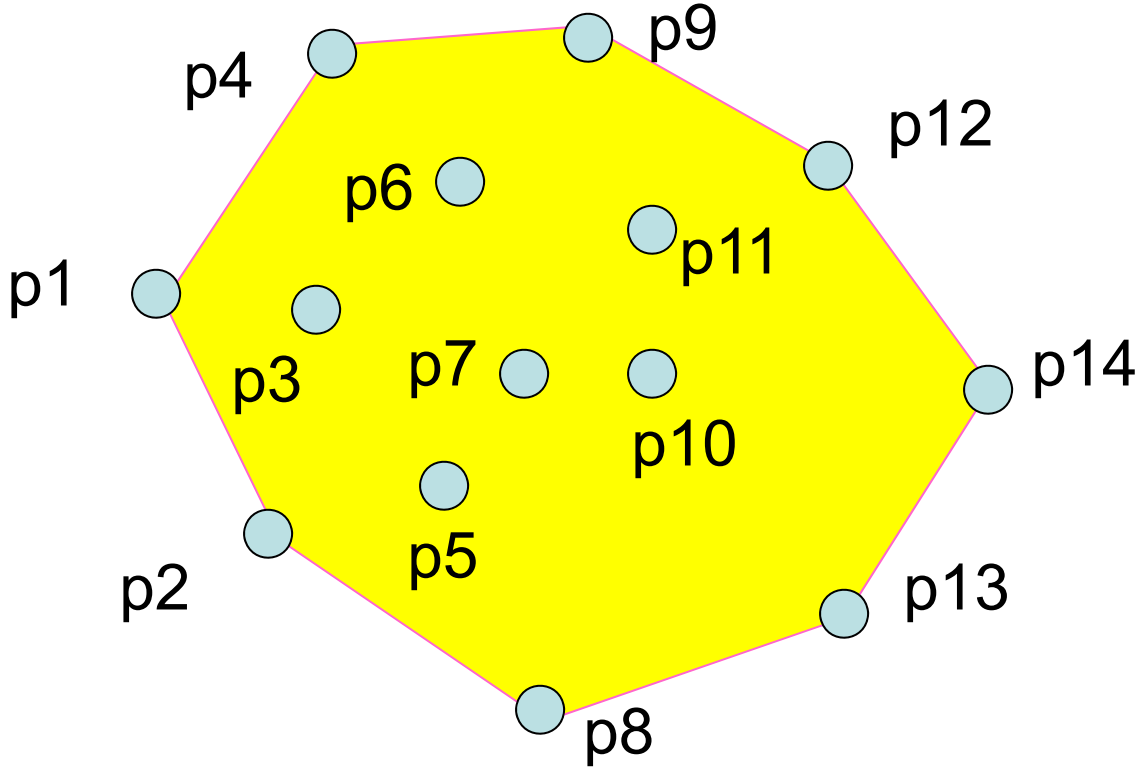
Characterization.

1. CH(S) is a convex polygon
2. Vertices of CH(S) are points of S
3. CH(S) contains all points of S



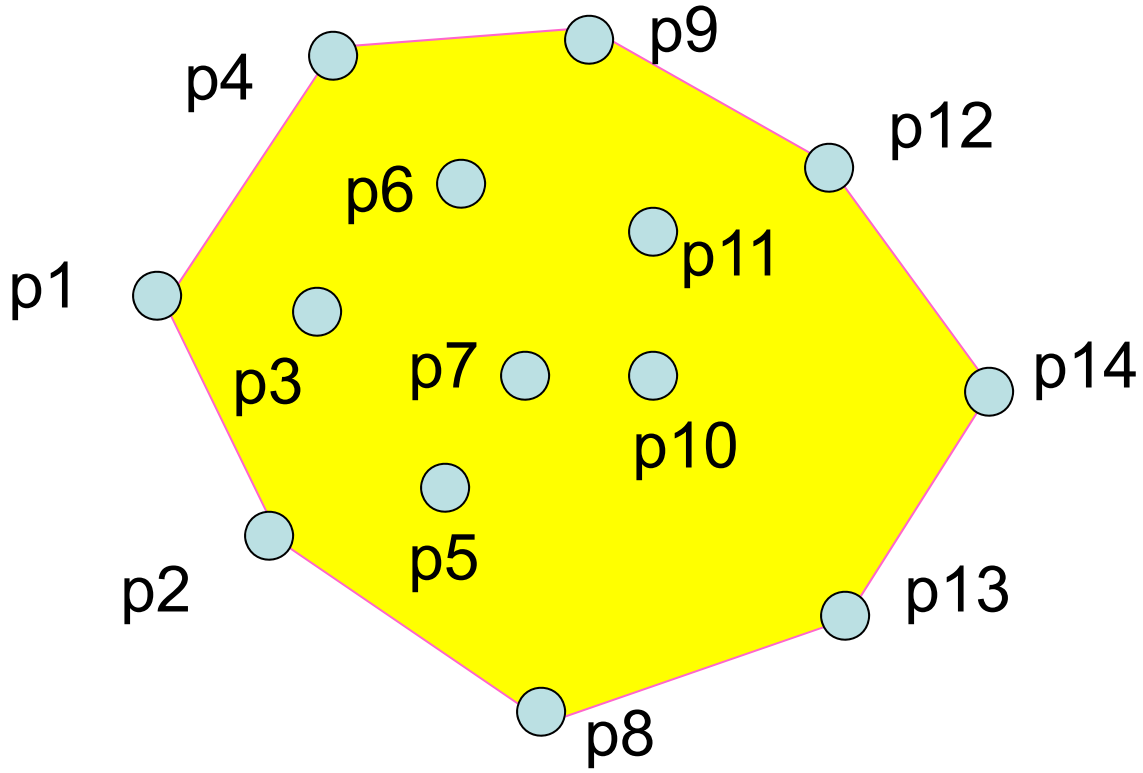
Homework: Prove it!

A representation of CH(S):
the list of vertices in a clockwise order starting from the leftmost one



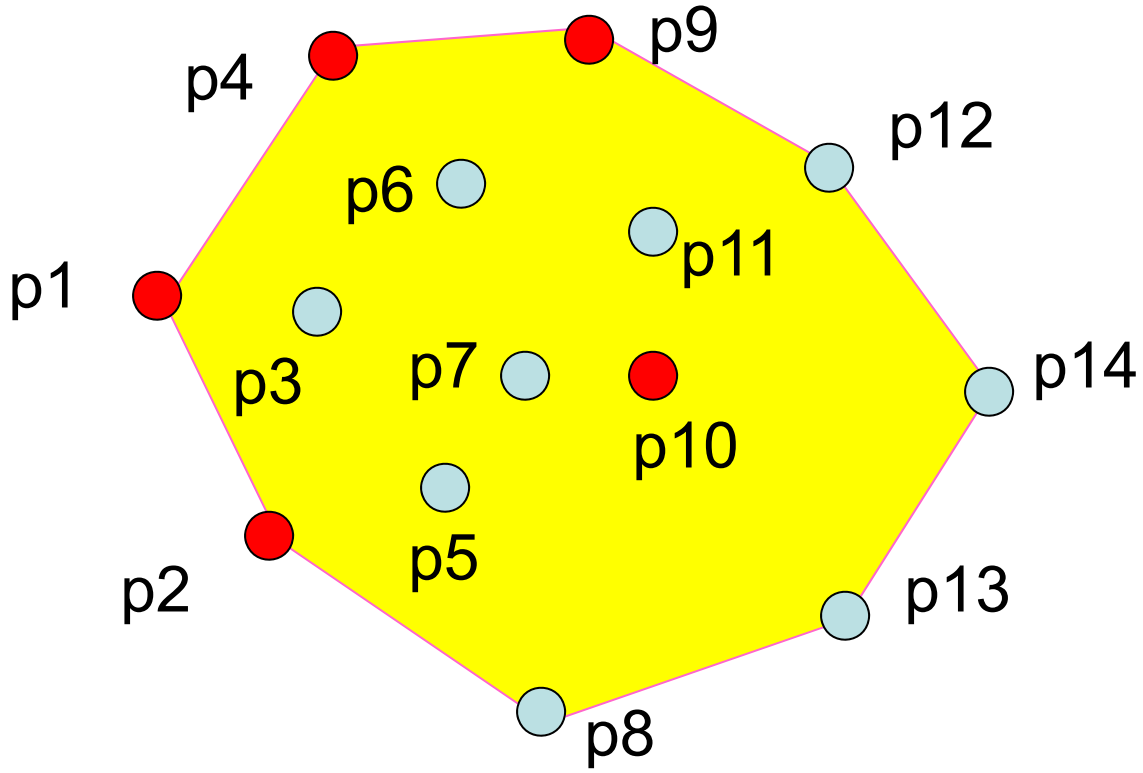
Question: Show CH(S) of the above picture in the above representation.

Now, the problem is in the discrete and finite world



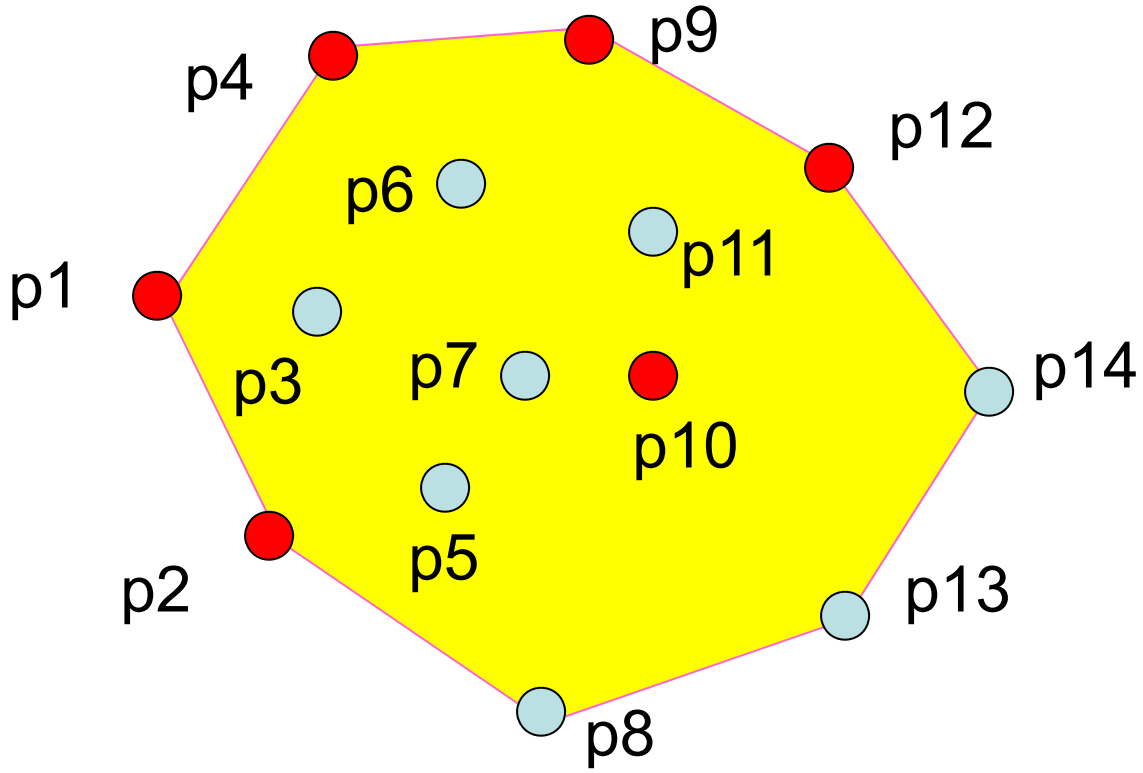
Find the (partial) permutation of S forming the convex hull.

Verification problem



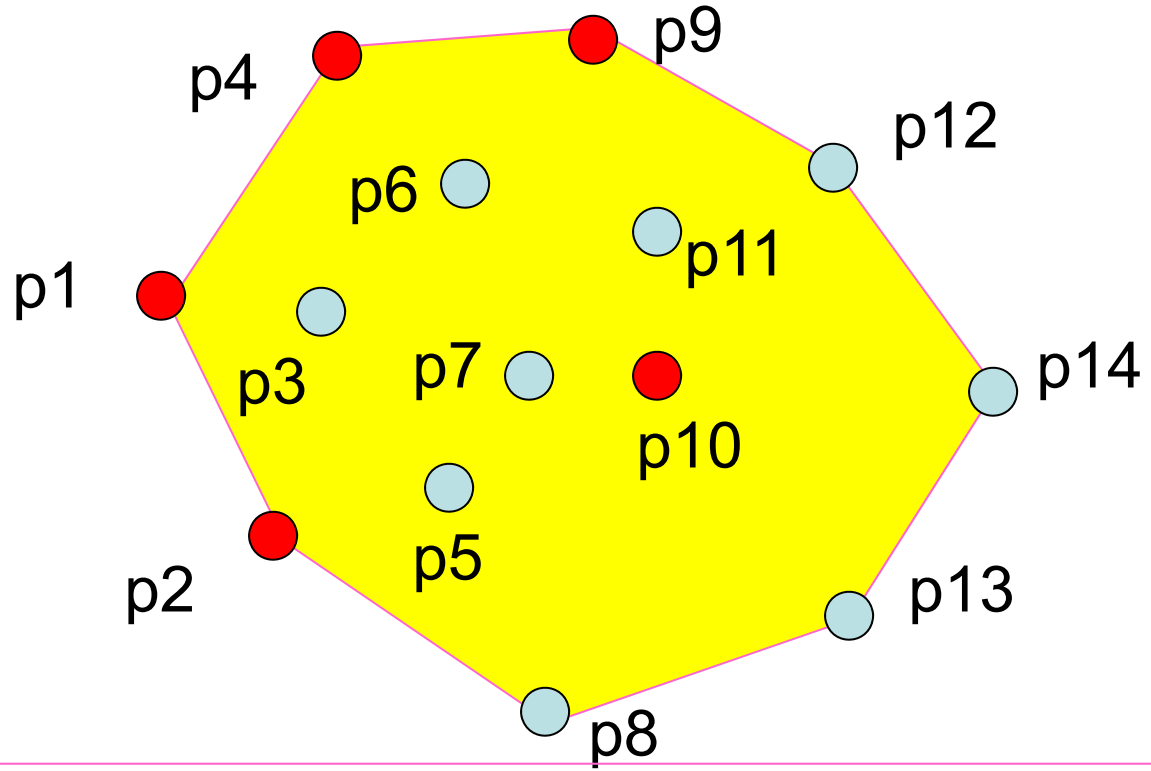
Given a list (p1, p2,p10,p12,p9,p4), verify whether it is CH(S).

Verification problem 1



Given a list (p1, p2,p10,p12,p9,p4), verify whether it gives a convex polygon

Verification problem 2



If the list (p1, p2,p10,p12,p9,p4) gives a convex polygon, show all other points are contained in it.

A brute-force algorithm

- Algorithm 1:
  - Generate all possible partial permutations of S
  - For each permutation P, verify it gives CH(S)
- Questions
  - Is the above algorithm always correct?
  - How much time does it take if n= 1000.
- Conclusion: We need a better algorithm.
- Question: Please consider a better algorithm than this!

Analysis of Algorithm

- Time Complexity
  - Given an input of size n (words/bits), how many basic steps are required in an algorithm?
    - Arithmetic operations
    - Comparisons, Data Access (read, write)
    - Floor/Ceiling [314,1592] = 314
  - T(n): number of basic steps
  - Asymptotic time complexity
    - $T(n) < c\, f(n)$ for a suitable constant c and a familiar function f(n)
    - We write $T(n) = O(f(n))$
- Classification of time complexity
  - Polynomial time algorithm: f(n) is a polynomial in n
    - Linear time algorithm: $T(n) = O(n)$
    - Quadratic time algorthm: $T(n) = O(n^2)$
  - Exponential time algorithm : e.g., $f(n) = 2^n$
  - Unbounded time algorithm : No such f(n)

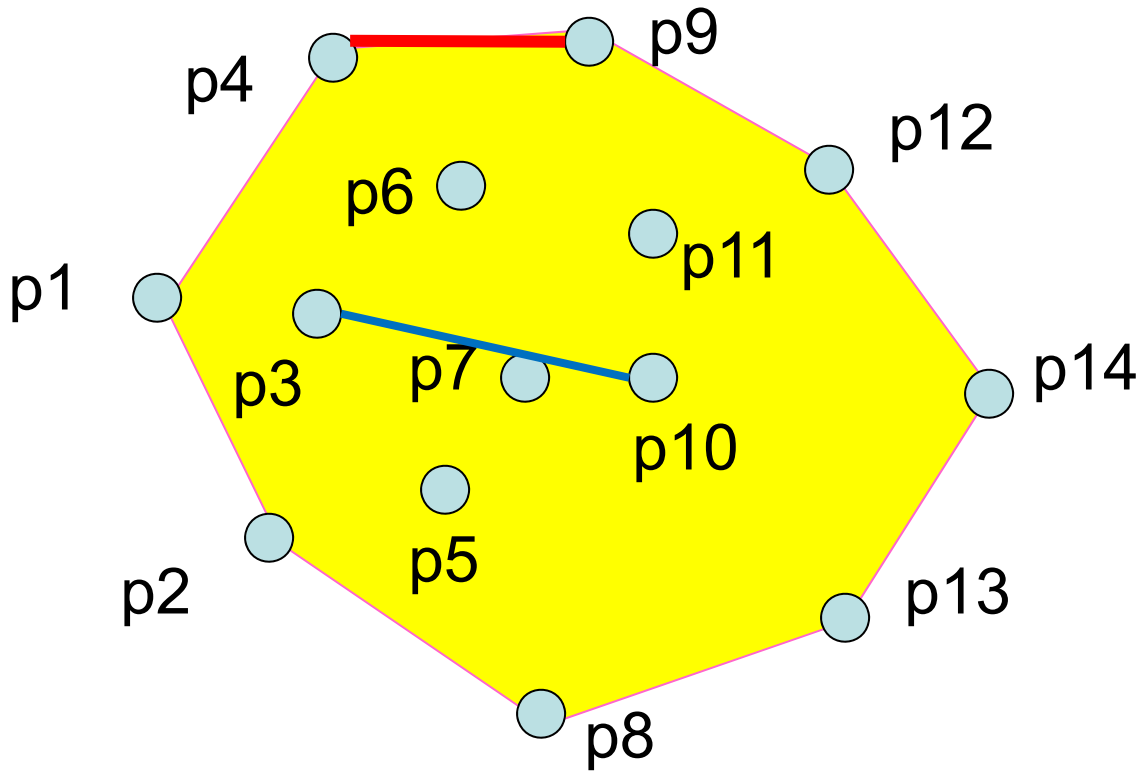
Complexity of a problem

- Complexity of a problem X
  - The complexity of X is O(f(n)) if there is an algorithm to solve X in O(f(n)) time
  - The complexity of X is Ω(f(n)) if there is no algorithm to solve X in o(f(n)) time
    - o(f(n)): strictly smaller than O(f(n))
  - The complexity of X is Θ（f（n)) if it is both O(f(n)) and Ω(f(n))
- Complexity class of a problem
  - A problem X is in class P if there is a polynomial time algorithm to solve X, that is, the complexity of X = O(f(n)) for a polynomial f(n).

Typical problems in class **P**

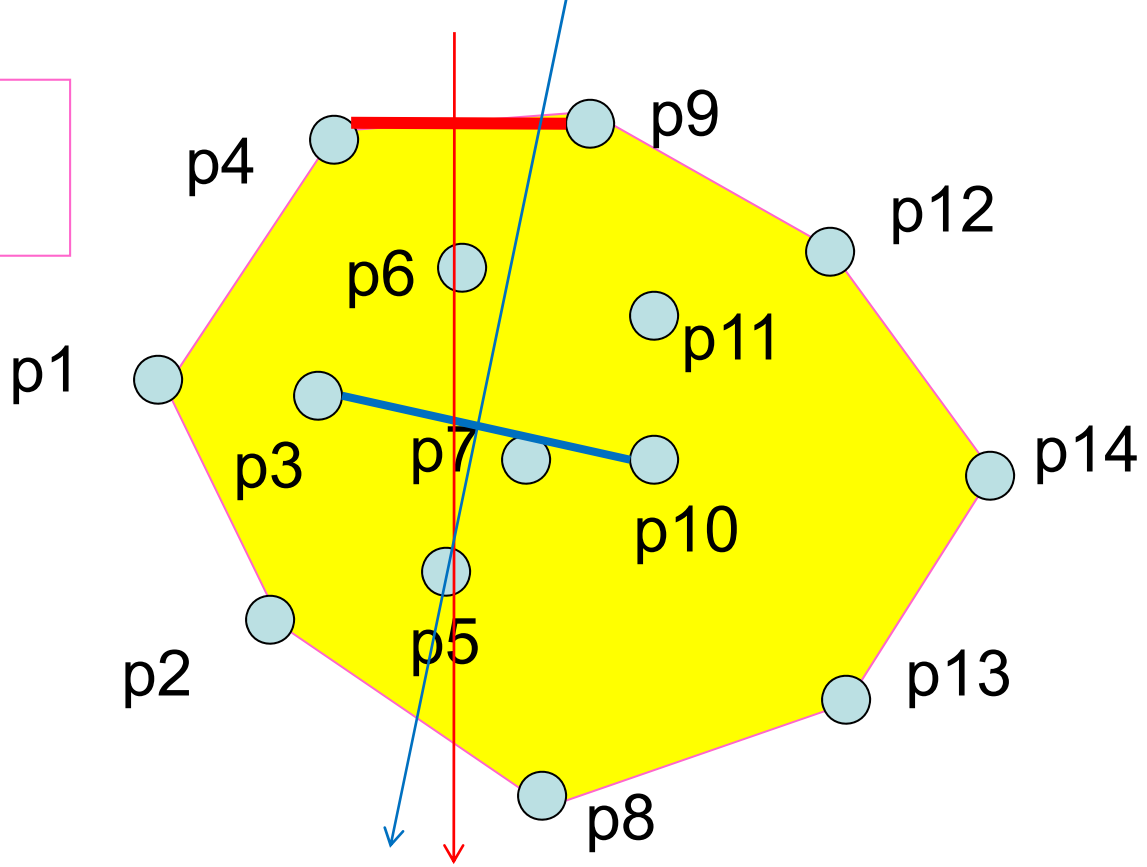
- Finding maximum element in a given set of n numbers : $\Theta(n)$ time
- Sorting n numbers: $O(n \log n)$ time
  - $\Theta(n \log n)$ if we restrict the computation model
- Computing the "distance" of two DNA sequences of length n: $O(n^2)$ time
- Computing convex hull of n points in the plane
  - How to do it? What is the time complexity?

Edge verification



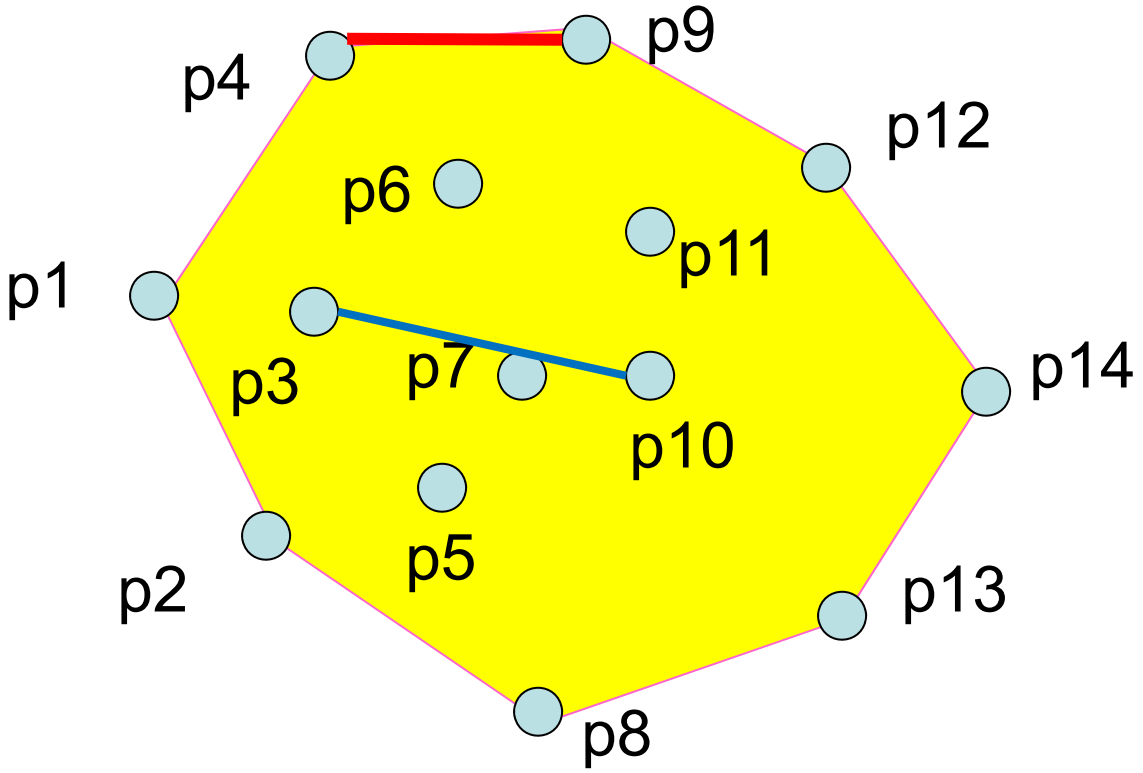
(p4, p9) is an edge of the convex hull, while (p3,p10) is not. How to distinguish them?

Edge verification



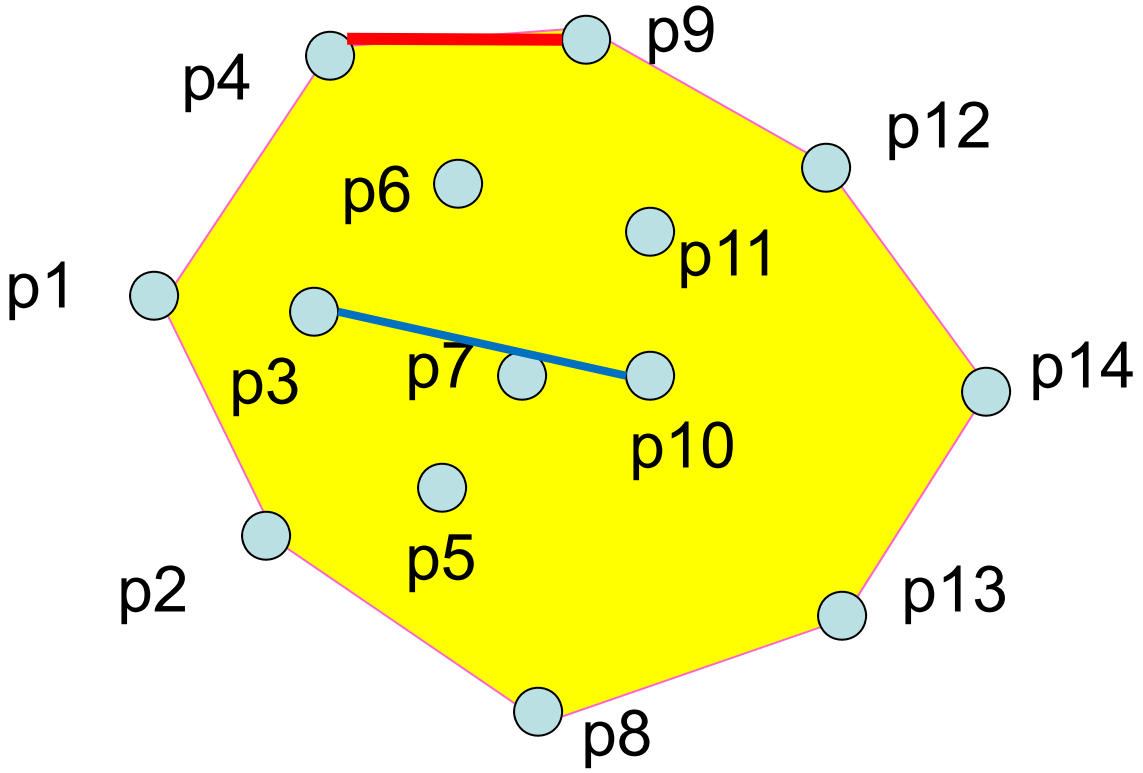
- Order all points in the orthogonal direction to the edge we want to verify
- Convex hull edge if and only if its endpoints are both maximum (or minimum) in the ordering

Convex hull algorithm using edge verification



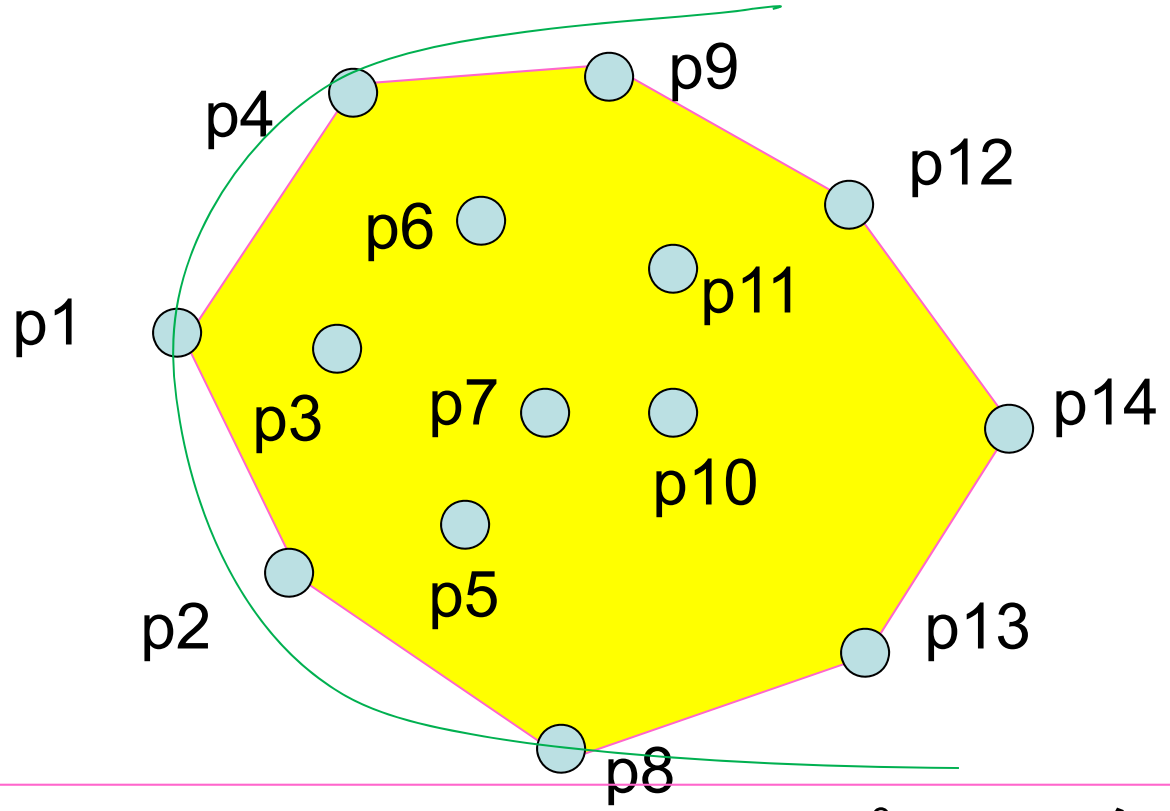
- Verify all n(n-1)/2 candidate edges
- Collect all the convex hull edges
- Arrange them into convex hull
  - How to do it?

Time complexity of the algorithm using edge verification



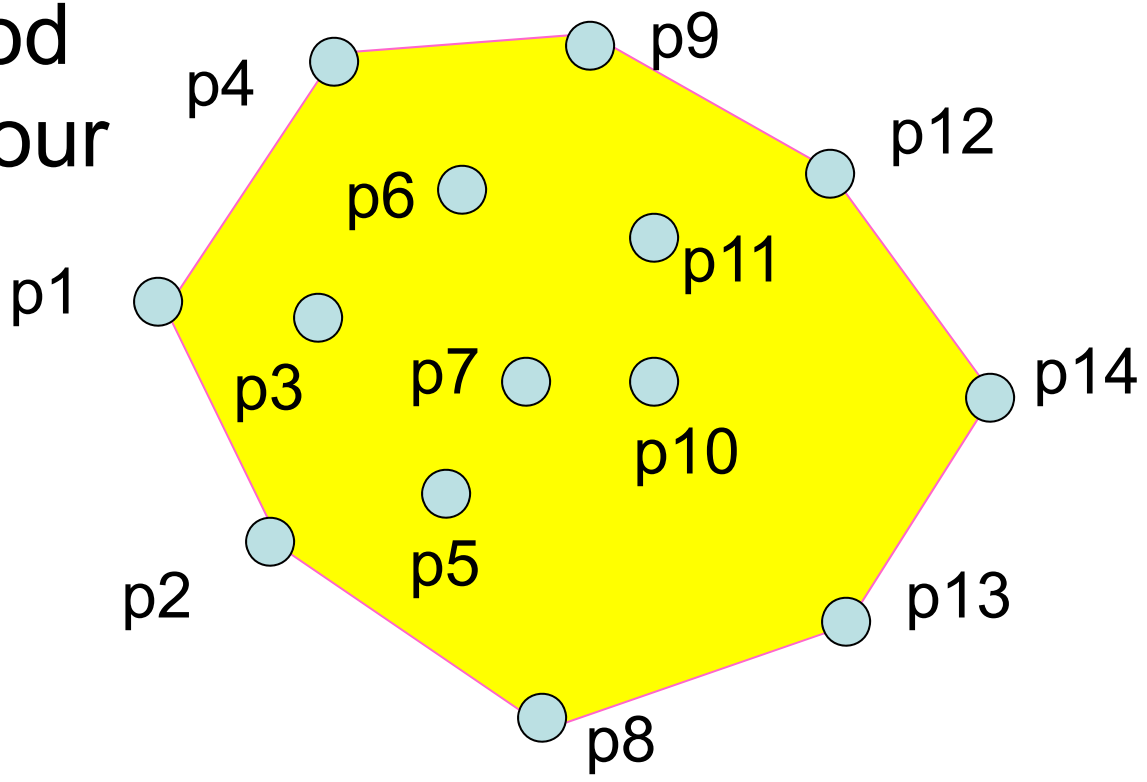
- $O(n^3)$ time algorithm
  - Edge verification = O(n) time for each candidate
  - $O(n^2)$ candidate edges
- Polynomial time. But very slow! How to improve?

Learn from our real life



- Consider points as pins on a board. (パチンコ台の釘だと思おう)
- You are given a string. How to realize the convex hull. (紐を使って凸包を計算しよう)
- “Gift wrapping” algorithm or……

Another method learning from our real life

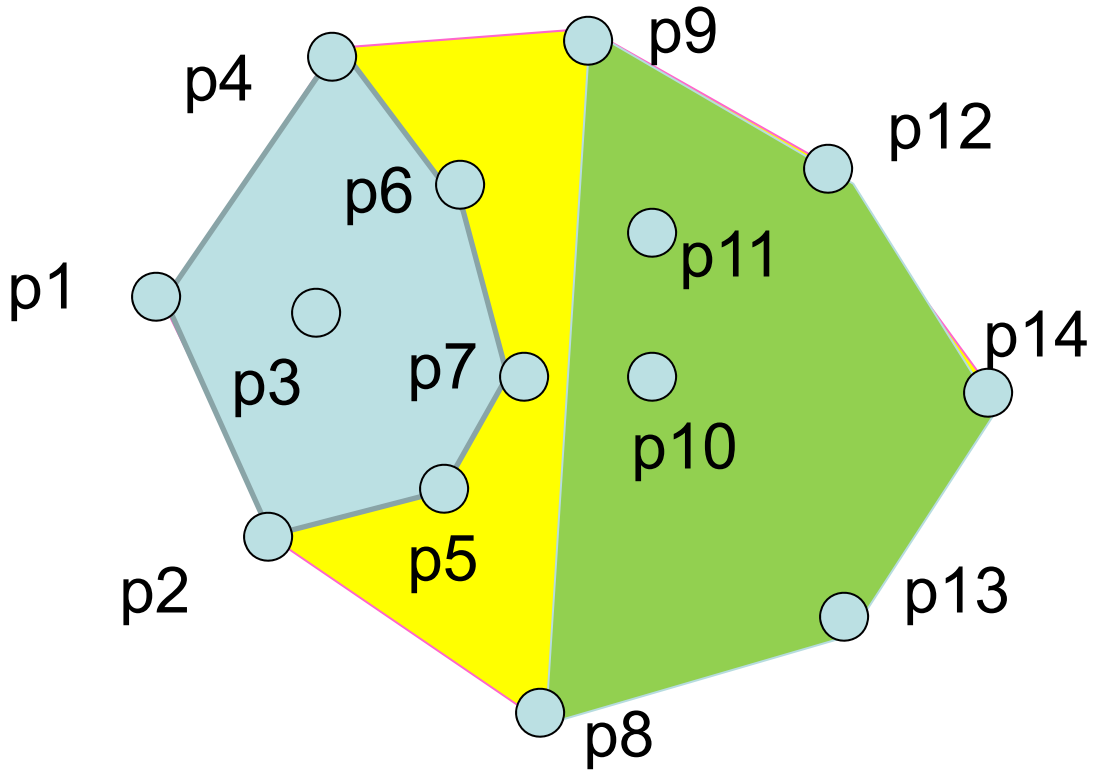


- Consider points as pins on a board.
- You can touch from any given direction by hand.

How to realize the convex hull.

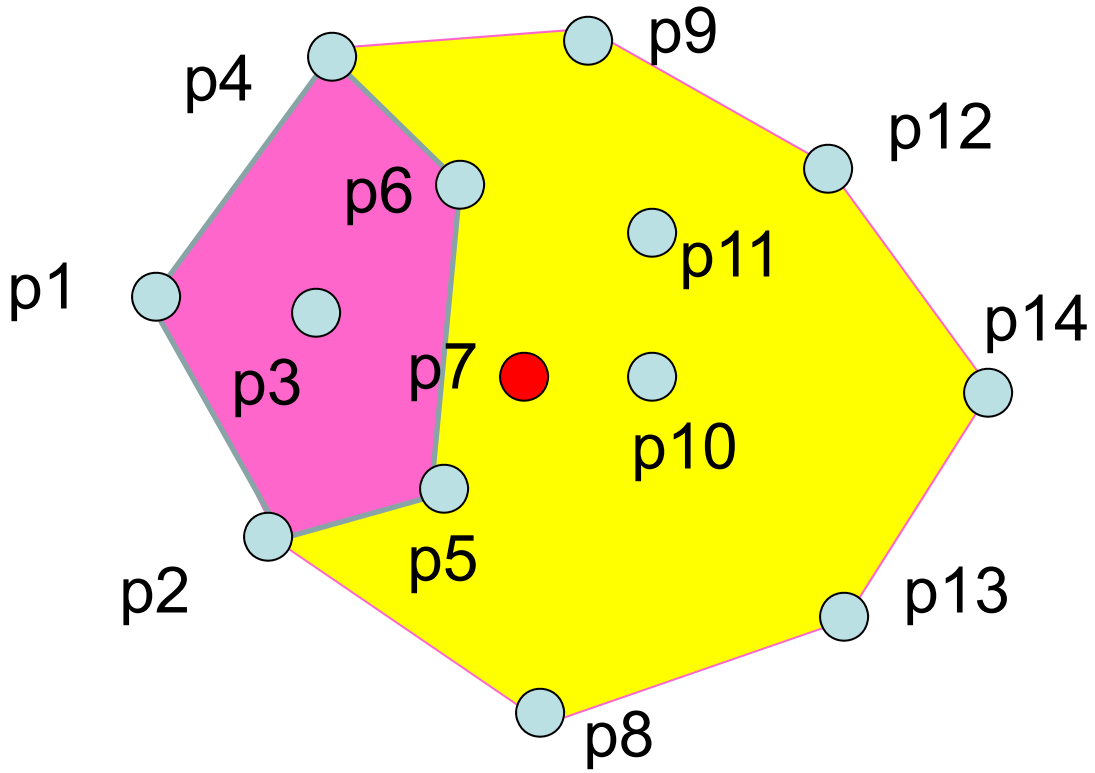
- “hand probing” algorithm

Algorithmic paradigms
1. Divide and conquer



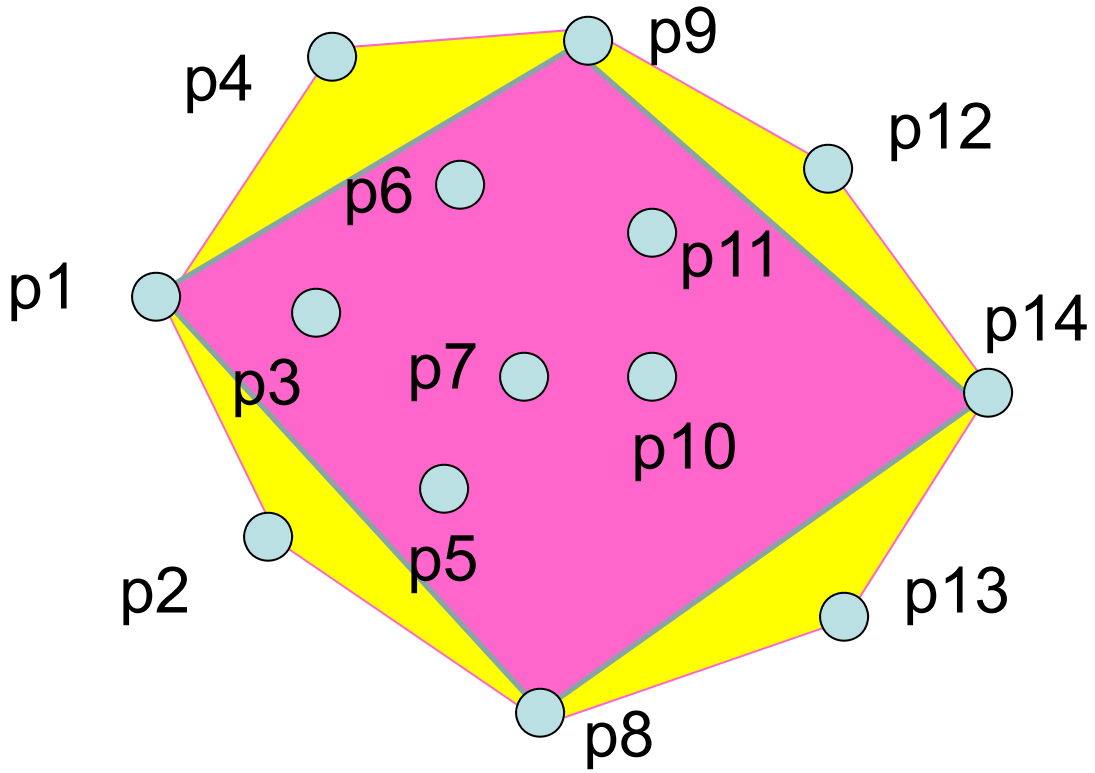
- Process “left half” and “right half” independently
- Merge two outputs

Algorithmic paradigms
2. Incremental method



- Process points left-to-right
- Start from the triangle formed by p1,p2,p3, and add points one by one updating the convex hull

Algorithmic paradigms
3. Prune by preprocessing



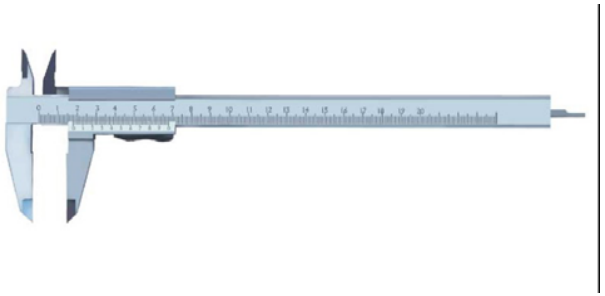
- Remove all points in the pink quadrangle
- Run an algorithm discussed before

Time complexities of algorithms

- Brute-force:
- Edge verification:
- Gift wrapping:
- Probing:
- Divide & Conquer:
- Left to right incremental:
- Pruning + Gift wrapping

Applications

- Classical use of convex hulls
  - Diameter computation
    - Rotating caliper
  - Fast collision detection



- Tokuyama's original use of convex hull algorithms
  - Statistics
  - Data mining (!)
  - Image processing (!!)