# 線形及び整数計画問題に対する主算法の計算量について

# The Complexity of Generic Primal Algorithms for Linear and Integer Programming Problems

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## References

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A. S. Schulz and R. Weismantel,
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K. Wayne,
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## Outline

- Optimization Problems and Primal Algorithms
- (Directed) Negative Circuit Problem
- Min Ratio Canceling Algorithm
- Relationship with Interior Point Methods
- Relaxed Version of Algorithm
- Application: Generalized Flow

○ ● ○ ● **Problem** ● ○ ● ○ — general LP, IP, or MIP

Input: $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$, $c \in \mathbf{R}^n$, $u \in (\mathbf{R} \cup \{+\infty\})^n$

| **(OPT)** | | |
|---|---|---|
| | minimize | $c^T x$ |
| | subject to | $Ax = b$, |
| | | $0 \leq x \leq u$, |
| | | $x \in \mathbf{R}^n$ |
| | | (or $x \in \mathbf{Z}^n$ or $x \in \mathbf{R}^{n_1} \times \mathbf{Z}^{n_2}$) |

$\mathcal{F} \equiv$ feasible region of (OPT)

**Primal Algorithms**

- Always keep a feasible solution
- Iteratively improve an objective function value

Examples

- primal simplex & interior point methods for LP
- augmenting path algo. for max flow
- negative cycle canceling algo. for min cost flow
- local search methods for NP-hard problems

## ○ ● ○ ● (Directed) Negative Circuit ● ○ ● ○

- $d \in \mathbf{R}^n$ : circuit $\overset{\text{def}}{\Longleftrightarrow}$ $Ad = 0$, $\{j \mid d(j) \neq 0\}$ is minimal

Can we improve a current solution $x$?

$\Updownarrow$

$\exists ? x' \in \mathcal{F}$ s.t. $c^T x' < c^T x$

$\Updownarrow$

$\exists ? d \in \mathbf{R}^n$ s.t. $c^T d < 0,\ Ad = 0,\ 0 \leq x + d \leq u$

**Negative Circuit Problem (NC)**

| | |
|---|---|
| Input: | $c \in \mathbf{R}^n$, $x \in \mathcal{F}$ |
| Output: | $d \in \mathbf{R}^n$ |
| | s.t. $c^T d < 0,\ Ad = 0,\ 0 \leq x + d \leq u$, |
| | or "There is no negative circuit" |

## ○ • Negative Circuit Canceling Algorithm • ○

**Step 0:** $x := 0$.

**Step 1:** Solve (NC) with $c, x$.

- ○ $\nexists$ negative circuit $\Longrightarrow$ Stop. $x$ is optimal.
- ○ $\exists$ negative circuit $d \Longrightarrow$ 'Cancel' $d$:

  $x := x + \alpha d \quad (\alpha = \sup\{\alpha' \mid 0 \leq x + \alpha' d \leq u\})$.

**Step 2:** Go to Step 1.

(NC) $\Longrightarrow$ (OPT) : not polynomial in general

$d^+$: $d^+(j) = \max\{d(j), 0\}$, positive part of $d$
$d^-$: $d^-(j) = \max\{-d(j), 0\}$, negative part of $d$

**Directed Negative Circuit Problem ($\mathrm{NC}_\pm$)**

——— Introduce costs for both $d^+$ and $d^-$

| | |
|---|---|
| Input: | $c', c'' \in \mathbf{R}^n$, $x \in \mathcal{F}$ |
| Output: | $d = d^+ - d^- \in \mathbf{R}^n$ |
| | s.t. $(c')^T d^+ + (c'')^T d^- < 0$, |
| | $Ad = 0,\ 0 \leq x + d \leq u$, |
| | or "There is no such $d$" |

Min cost flow case:
($\mathrm{NC}_\pm$) $\Longleftrightarrow$ negative circuit detection in residual graph

$(\mathrm{NC}_{\pm}) \Longrightarrow (\mathrm{OPT})$ : Polynomial ? — Yes!

We show

$\boxed{(\mathrm{NC}_{\pm})}$

poly. time

$\boxed{\boxed{\text{min ratio circuit (MRC)}}}$ poly. time

poly. time

$\boxed{(\mathrm{OPT})}$

## ○ ● ○ ● Minimum Ratio Circuit Problem ● ○ ● ○

Input: $c', c'' \in \mathbf{R}^n$, $w', w'' \in \mathbf{R}^n_+$, $x \in \mathcal{F}$

| | | |
|---|---|---|
| **(MRC)** | minimize | $\dfrac{(c')^T d^+ + (c'')^T d^-}{(w')^T d^+ + (w'')^T d^-}$ |
| | subject to | $Ad = 0$, $d = d^+ - d^- \in \mathbf{R}^n$, |
| | | if $x(j) = 0$ then $d(j) \geq 0$, |
| | | if $x(j) = u(j)$ then $d(j) \leq 0$. |

Min cost flow case:
(MRC) $\Longleftrightarrow$ finding min ratio circuit in residual graph

($\mathrm{NC}_\pm$) $\Longrightarrow$ (MRC): (Weakly) Polynomial!!

## ○ ● ○ ● Proof for polynomiality ● ○ ● ○

$$\frac{(c')^T d^+ + (c'')^T d^-}{(w')^T d^+ + (w'')^T d^-} \leq \lambda$$

$$\Longleftrightarrow \ (c' - \lambda w')^T d^+ + (c'' - \lambda w'')^T d^- \leq 0$$

$$f(\lambda) \equiv \min_d \{(c' - \lambda w')^T d^+ + (c'' - \lambda w'')^T d^-\}$$

**Theorem.**

- $f(\lambda)$: concave function,
- $f(\lambda_*) = \dfrac{(c')^T d_*^+ + (c'')^T d_*^-}{(w')^T d_*^+ + (w'')^T d_*^-} = 0$

$\Longrightarrow \lambda_*, d_*$: optimal for (MRC)

$\Longrightarrow$ binary search with ($\mathrm{NC}_\pm$) $\longrightarrow$ weakly polynomial

## ○ • Min Ratio Circuit Canceling Algorithm • ○

— cancel the following min ratio circuit iteratively

$$\begin{array}{ll} \text{minimize} & \dfrac{c^T d}{\displaystyle\sum_{j=1}^{n} \frac{1}{u(j)-x(j)} d^+(j) + \sum_{j=1}^{n} \frac{1}{x(j)} d^-(j)} \\ \text{subject to} & Ad = 0,\ d = d^+ - d^- \in \mathbf{R}^n, \\ & \text{if } x(j) = 0 \text{ then } d(j) \geq 0, \\ & \text{if } x(j) = u(j) \text{ then } d(j) \leq 0. \end{array}$$

**Algorithm MinRat**

**Step 0:** $x := 0$.

**Step 1:** $\nexists$ negative circuit $\Longrightarrow$ Stop. $x$: optimal.

**Step 2:** Find a min ratio circuit.

**Step 3:** Cancel the min ratio circuit.

**Step 4:** Go to Step 1.

$$w_x(d) \equiv \sum_{j=1}^{n} \frac{1}{u(j) - x(j)} d^+(j) + \sum_{j=1}^{n} \frac{1}{x(j)} d^-(j)$$

**Lemma.**
$x$, $x'$: current and next feas. sol., $x^*$: opt. sol.
$d$: current min ratio circuit

(i) $c^T(x' - x) \leq \frac{1}{m} c^T(x^* - x)$, (ii) $\frac{c^T d}{w_x(d)} \leq \frac{c^T(x^* - x)}{m}$.

**Proof:** $x' - x = \alpha d \Longrightarrow$

$$\frac{c^T(x' - x)}{w_x(x' - x)} = \frac{c^T d}{w_x(d)} \leq \frac{c^T(x^* - x)}{w_x(x^* - x)}. \tag{1}$$

$\forall j,\ -x(j) \leq x^*(j) - x(j) \leq u(j) - x(j) \Longrightarrow$

$$w_x(x^* - x) \leq m. \tag{2}$$

$\exists j$ s.t. $\alpha d(j) = u(j) - x(j)$ or $-\alpha d(j) = x(j) \Longrightarrow$

$$w_x(x' - x) = w_x(\alpha d) \geq 1. \tag{3}$$

$$
\begin{aligned}
B &\equiv \max_{i,j}\{\text{bit length of } a(i,j)\},\\
U &\equiv \max_{j}\{\text{bit length of } u(j)\},\\
C &\equiv \max_{j}\{\text{bit length of } c(j)\}.
\end{aligned}
$$

**Theorem.**
Algorithm **MinRat** finds an optimal solution in $O(n^4\langle B\rangle + n^3\langle U\rangle + n^2\langle C\rangle)$ iterations.

- Find $\tilde{x} \in \mathcal{F}$ with $c^T(\tilde{x} - x^*) < \epsilon$ by **MinRat**
  $(\epsilon \equiv 2^{-(4n^3\langle B\rangle + 4n^2\langle U\rangle + n\langle C\rangle)})$
- Round $\tilde{x}$ to an optimal extreme vector $\widehat{x}$
  by canceling at most $n$ circuits

## ○ How to Round a Nearly Optimal Solution •

**Lemma.**
$x \in \mathcal{F}$: extreme vector of $\mathcal{F}$
$\iff$ $\nexists$ circuit $d \in \mathbf{R}^n$ with
$(*)$ $\quad \{j \mid d(j) \neq 0\} \subseteq \{j \mid 0 < x(j) < u(j)\}$.

For nearly opt. sol. $\tilde{x}$, repeat the following:

- Find a circuit $d \in \mathbf{R}^n$ with $(*)$, $c^T d \leq 0$.
- Cancel $d$.

$\Longrightarrow$ optimal extreme vector $\widehat{x}$

$\{j \mid 0 < x(j) < u(j)\}$ decreases in every iteration
$\Longrightarrow$ $\sharp$ (iteration) $\leq n$

## ○ • History of Min Ratio Circuit Canceling • ○

☆ Zimmermann (80 $\sim$ 90) —
- study on projective methods (IPM)
- comb. opt. with "unusual" obj. func.
- negative cycle method for (submodular) flows

☆ Wallacher (1989) — MinRat for network flow

☆ Wallacher & Zimmermann (1994)
— MinRat for submodular flow

★ McCormick & Shioura (1996)
— MinRat for { dual network flow / LP over unimodular linear space

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◇ Schulz, Weismantel & Ziegler (1995)
— min mean canceling (= MinRat) for 0-1 IP

◇ Schulz &, Weismantel (1998) — MinRat for general IP

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◆ Wayne (1999) — MinRat for generalized network flow

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★ McCormick & Shioura (1999) — MinRat for general LP

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★ McCormick, Schulz, Shioura & Weismantel (1999)
— MinRat for MIP

## ○ • Relationship with Interior Point Method • ○

★ minimize the ratio

$\Longrightarrow$ small $|d(j)|$ if $u(j) - x(j)$, $x(j)$ small

$\Longrightarrow$ stay away from the boundary of $\mathcal{F}$

★ min ratio circuit $\simeq$ affine scaling direction

standard form LP

| | |
|---|---|
| minimize | $c^T x$ |
| subject to | $Ax = b,\ x \geq 0.$ |

◎ (primal) affine scaling direction

$$\text{min. } c^T d \quad \text{sub. to } Ad = 0,\ \sqrt{\sum_{j=1}^{n} \left(\frac{d(j)}{x(j)}\right)^2} = 1.$$

$$\text{min. } \frac{c^T d}{\sqrt{\sum_{j=1}^{n} \left(\frac{d(j)}{x(j)}\right)^2}} \text{ sub. to } Ad = 0.$$

◎ min ratio circuit

$$\text{min.} \quad \frac{c^T d}{\sum_j \frac{1}{x(j)} d^-(j)} = \frac{c^T d}{\sum_{j:d(j)<0} \left| \frac{d(j)}{x(j)} \right|}$$

$$\text{sub. to} \quad Ad = 0.$$

◎ affine scaling direction w.r.t. $L_1$-norm

$$\text{min.} \quad \frac{c^T d}{\sum_{j=1}^{n} \left| \frac{d(j)}{x(j)} \right|} \quad \text{sub. to } Ad = 0.$$

- min ratio circuit
  — linear convergence globally ○ $\Rightarrow$ polynomial
- affine scaling direction
  — linear convergence locally ○ globally ??

## ○ ● ○ ● Relaxed Version of MinRat ● ○ ● ○

min ratio circuit $\longrightarrow$ circuit with small ratio

**Algorithm RMinRat**

- $\mu$-phase ($\mu < 0$) — repeat the following:
  - ○ find a circuit with ratio $\leq \mu$
  - ○ cancel the circuit
- no such circuit $\Longrightarrow \mu := \mu/2$

**Lemma.** At most $2n$ iterations in a $\mu$-phase.

**Lemma.** $\sharp$ ($\mu$-phase) $= O(n^3\langle B\rangle + n^2\langle U\rangle + n\langle C\rangle)$.

$\Longrightarrow$ $\sharp$ (iter. of **RMinRat**) $=$ $\sharp$ (iter. of **MinRat**)

- detection of a circuit with ratio $\leq \mu \longrightarrow$ (NC)

$\Longrightarrow$ cost in each iteration:

**RMinRat** (NC) $\leq$ **MinRat** (MRC)

## ○ ● ○ ● Application: Generalized Flow ● ○ ● ○

generalized flow — network flow with 'gain'

Input: directed graph $G = (V, E)$,
upper bound $u(i,j) \in \mathbf{R} \cup \{+\infty\}$, gain $\gamma(i,j) \in \mathbf{R}$

$$\begin{aligned}
&\text{min.} && \sum_{(i,j)\in E} c(i,j)x(i,j) \\
&\text{sub. to} && \sum_{(i,k)\in E} x(i,k) - \sum_{(k,i)\in E} \gamma(k,i)x(k,i) = b(i) \ \ (i \in V), \\
& && 0 \le x(i,j) \le u(i,j) \ \ ((i,j) \in E).
\end{aligned}$$

**MinRat** for generalized flow (Wayne 1999)
— first combinatorial polynomial algo.

## ○ ● ○ ● Detection of a Negative Circuit ● ○ ● ○

**(NC)** $\exists x$ : circuit s.t. $c^T x < 0$

Farkas Lemma

**(2VPI)** $\nexists \pi \in \mathbf{R}^V$ s.t.
$c(i,j) + \pi(i) - \gamma(i,j)\pi(j) \geq 0 \quad (\forall (i,j) \in E)$

(2VPI) — Two Variables Per Inequality system

- $\exists$ polynomial-time algo. for (2VPI)
- returns certificate (= nega. circuit) if no feas. sol.