Substitutes and Complements in Network Flows Viewed as Discrete Convexity

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Overview of Our Result

Max weight circulation problem

$$\text{Max.}\ \ w^T\xi\ \text{ sub. to. }\ N\xi = \mathbf{0},\ \ \mathbf{0} \le \xi \le c$$

- N: vertex-arc incidence matrix of digraph G=(V,A)
- w(a): weight of a∈A, c(a): capacity of a∈A

Our Interest:

combinatorial properties of optimal value function

$$F^*(w,c) \equiv \text{optimal value of max. weight circ. prob.}$$

function in problem parameters: weight w & capacity c

Overview of Our Result

- Gale-Politof (1981)
  - F*is submodular/supermodular w.r.t. some parameters
- well-known result in parametric LP
  - F* is convex/concave w.r.t. some parameters

- Our Result:
  - provide a better understanding from the viewpoint of "discrete convex analysis" (Murota 1996)
  - F* is M-convex/L-convex & M-concave/L-concave
  - nice properties in Mathematical Economics

Sub-/Supermodularity of F*

Theorem (Gale-Politof 1981)

P: "parallel" arc set, S: "series" arc set

F* is

(1) submodular w.r.t. {w(a) | a∈P} and {c(a) | a∈P}

(2) supermodular w.r.t. {w(a) | a∈S} and {c(a) | a∈S}

$f : \mathbf{R}^n \to \mathbf{R}$ is submodular

$$\iff f(x) + f(y) \geq f(x \vee y) + f(x \wedge y) \ (\forall x, y \in \mathbf{R}^n)$$

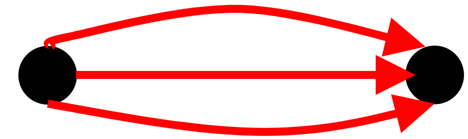
$$(x \vee y)_i = \max(x_i, y_i)$$

$$(x \wedge y)_i = \min(x_i, y_i)$$

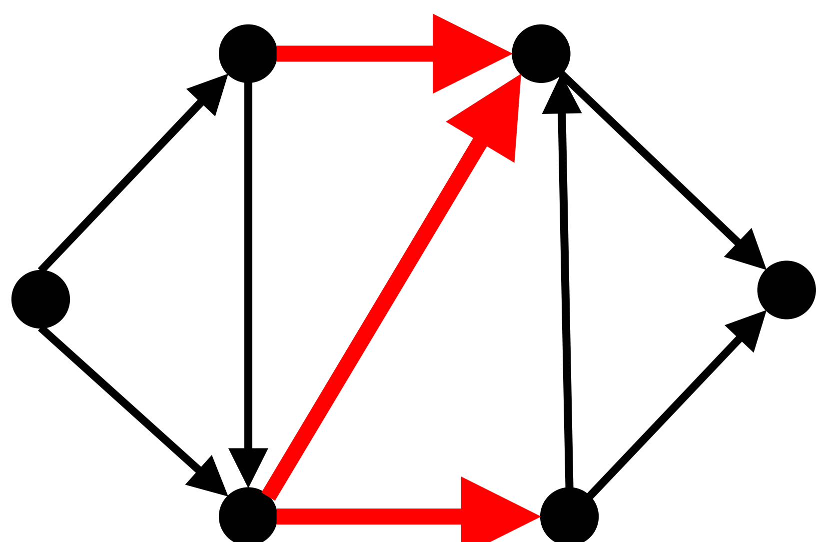
f is supermodular $\iff$ $-f$ is submodular

“Parallel” & “Series” Arcs

- $P \subseteq A$: a “parallel” arc set



$\Longleftrightarrow$ $\forall C$: undirected simple cycle in G, arcs in $P \cap C$ have different directions in C



“parallel” arc set

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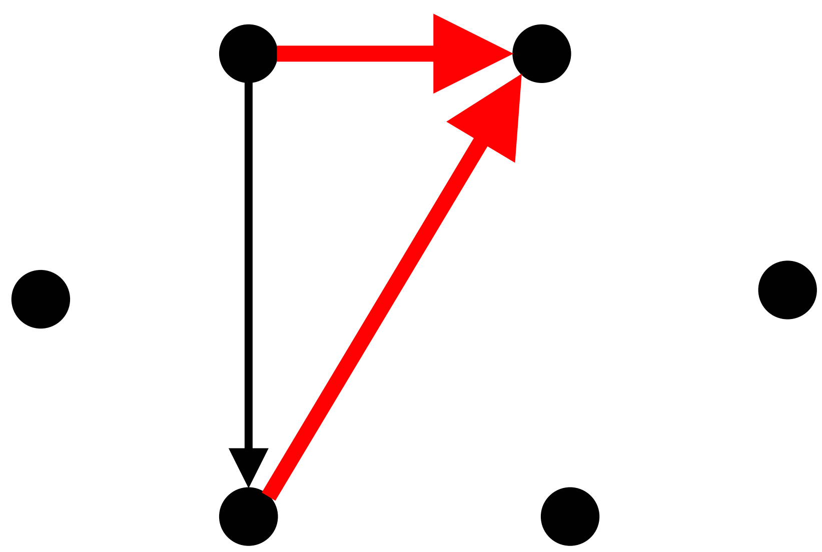

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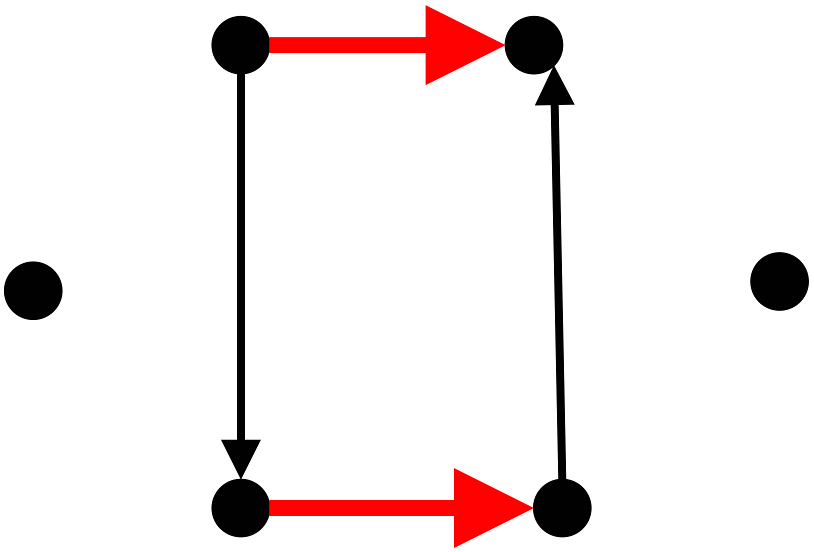

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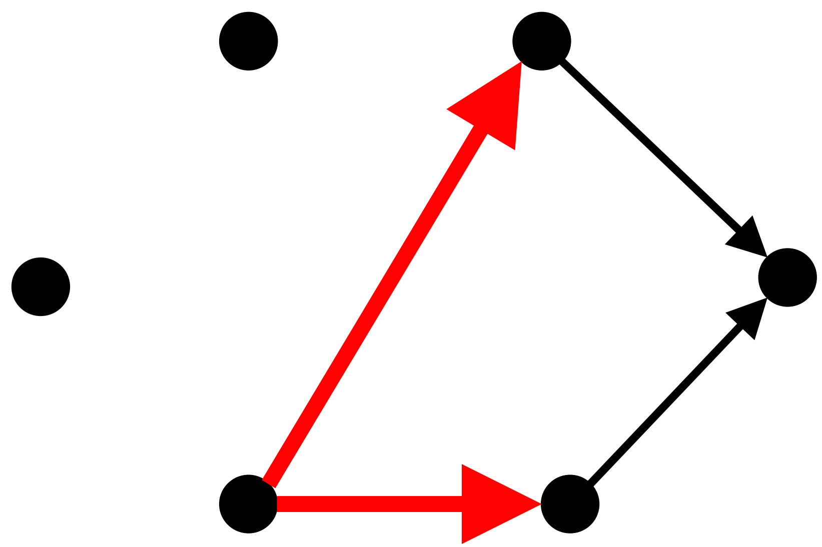
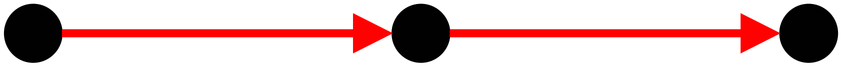
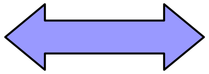


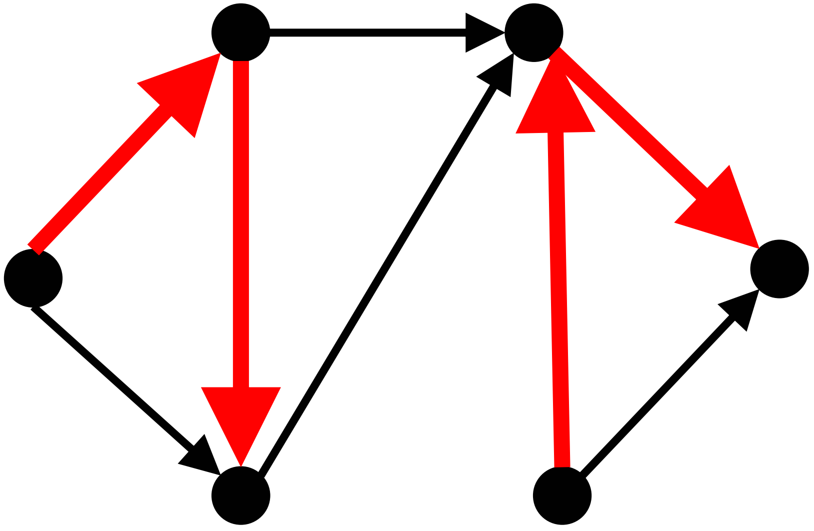
“Parallel” & “Series” Arcs

- S⊆A: a “series” arc set





∀C: undirected simple cycle in G,
arcs in S∩C have the same direction in C

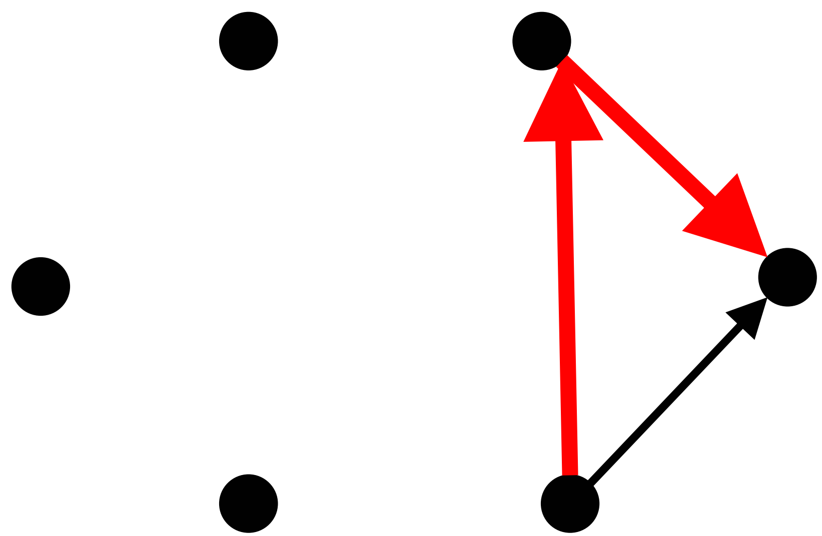


“series” arc set

“Parallel” & “Series” Arcs

- S⊆A: a “series” arc set

⟺ ∀C: undirected simple cycle in G, arcs in S∩C have the same direction in C



“Parallel” & “Series” Arcs

- S⊆A: a “series” arc set

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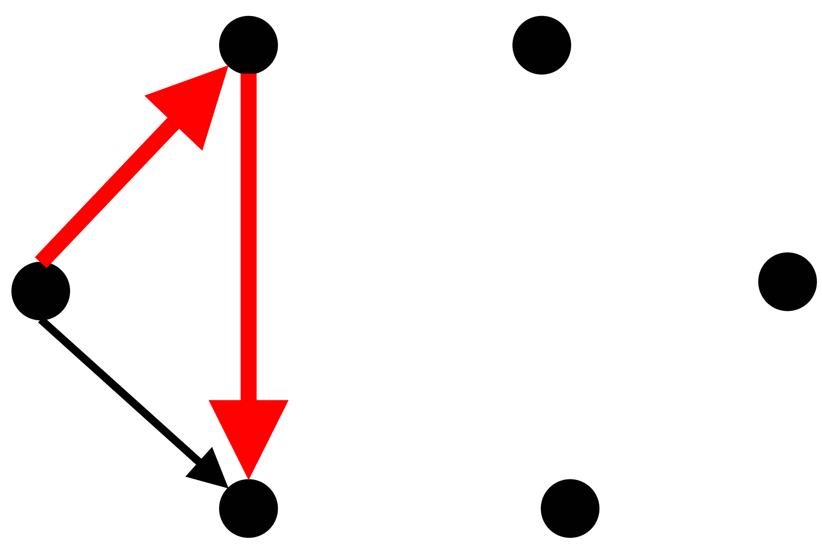


Sub-/Supermodularity of F*

Theorem (Gale-Politof 1981)

P: "parallel" arc set, S: "series" arc set

F* is

(1) submodular w.r.t. $\{w(a) \mid a \in P\}$ and $\{c(a) \mid a \in P\}$

(2) supermodular w.r.t. $\{w(a) \mid a \in S\}$ and $\{c(a) \mid a \in S\}$

Convexity/Concavity of F*

- well-known result in parametric LP

Proposition

F* is

(1) convex w.r.t. $\{w(a) \mid a \in A\}$
(2) concave w.r.t. $\{c(a) \mid a \in A\}$

(A: set of all arcs in G)

Summary of Previous Results

- P: “parallel” arc set,   S: “series” arc set

- F* is submodular & convex w.r.t. {w(a) | a∈P}
- F* is submodular & concave w.r.t. {c(a) | a∈P}
- F* is supermodular & convex w.r.t. {w(a) | a∈S}
- F* is supermodular & concave w.r.t. {c(a) | a∈S}

all combination of submodularity/supermodularity and convexity/concavity

Our Result

- P: "parallel" arc set,   S: "series" arc set

- F* is submodular & convex w.r.t. $\{w(a) \mid a \in P\}$
- F* is submodular & concave w.r.t. $\{c(a) \mid a \in P\}$
- F* is supermodular & convex w.r.t. $\{w(a) \mid a \in S\}$
- F* is supermodular & concave w.r.t. $\{c(a) \mid a \in S\}$

consequence of discrete convexity/concavity:
M-convexity/M-concavity
and L-convexity/L-concavity

M-convexity & L-convexity

- M-convex and L-convex functions
  - introduced by Murota (1996)
  - play central role in theory of “discrete convex analysis”
    - theoretical framework for well-solved combinatorial optimization problems
    - deep relationship with (poly)matroid theory
  - concepts for function over integer lattice Z^n

  → later extended to functions over real space R^n
  - enjoy nice properties as “discrete convexity”

Fundamental Properties of M-convexity & L-convexity

- M-convex, L-convex
  ➔ (can be extended to) ordinary convex functions
- M-convex ➔ supermodular
- L-convex ➔ submodular
- nice properties in Mathematical Economics

Our Main Result

Theorem

P: "parallel" arc set, S: "series" arc set

F^* is

(1) L-convex w.r.t. $\{w(a) \mid a \in P\}$

→ submodular & convex

(2) M-concave w.r.t. $\{c(a) \mid a \in P\}$

→ submodular & concave

(3) M-convex w.r.t. $\{w(a) \mid a \in S\}$

→ supermodular & convex

(4) L-concave w.r.t. $\{c(a) \mid a \in S\}$

→ supermodular & concave

Implication in Math Economics

- suppose F* represents utility function w.r.t. some goods
  - F*: submodular ⇔ goods are substitutes
    e.g., coffee and tea
  - F*: supermodular ⇔ goods are complements
    e.g., lock and key

  Gale Politof (1981)

- M-concavity → gross substitutes property
  single improvement condition
  no complementarity condition

F* has nicer property than sub-/supermodularity

Definition of M-convex Function

f is M-convex $\Longleftrightarrow$

$\forall x, y \in \mathbf{R}^n$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists \alpha_0 > 0$:

(i) $f(x) + f(y) \geq f(x - \alpha\chi_i) + f(y + \alpha\chi_i)$ $(0 \leq \forall\alpha \leq \alpha_0)$

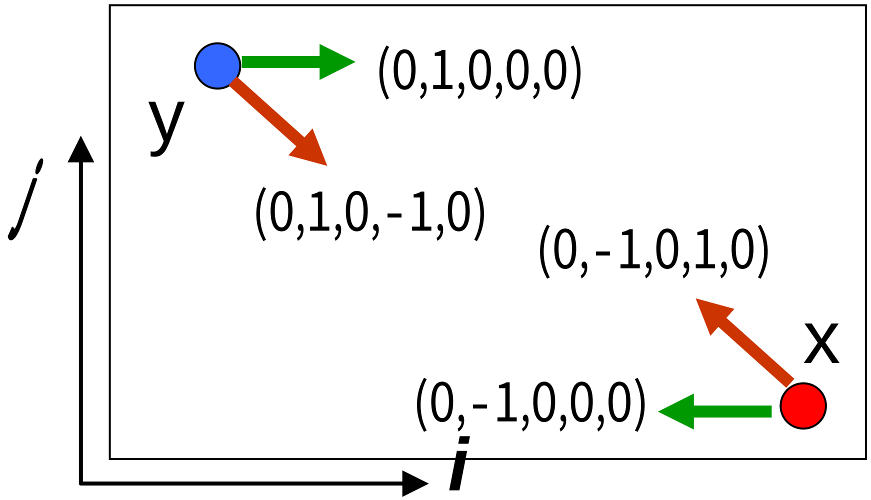
or (ii) $\exists j \in \mathrm{supp}^-(x-y)$ s.t.

$f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j))$ $(0 \leq \forall\alpha \leq \alpha_0)$

$\mathrm{supp}^+(x-y) = \{i \mid x(i) > y(i)\}$

$\mathrm{supp}^-(x-y) = \{i \mid x(i) < y(i)\}$

χ_i: i-th unit vector

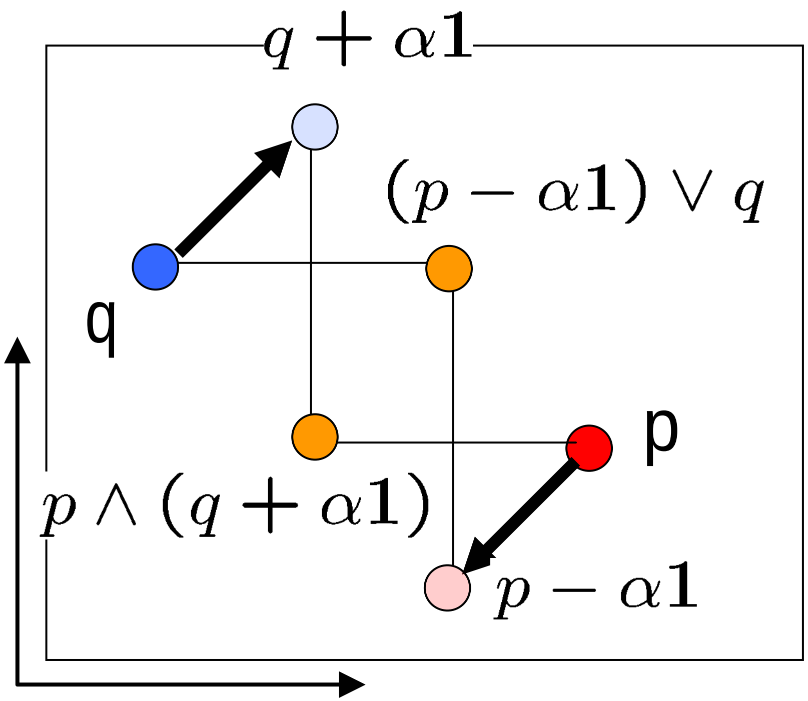


Definition of L-convex Function

g is L-convex $\Longleftrightarrow$

$$\boxed{g(p) + g(q) \geq g((p - \alpha\mathbf{1}) \vee q) + g(p \wedge (q + \alpha\mathbf{1})) \ \ (\forall p, q, \forall \alpha > 0)}$$

$$\mathbf{1} = (1, 1, \ldots, 1)$$



Extension to Linear Program

LP over Unimodular Linear Space

$$F^{\mathsf{LP}}(w, c) \equiv \mathsf{max}\{w^T \xi \mid N \xi = \mathbf{0},\ \ \mathbf{0} \leq \xi \leq c\}$$

totally unimodular matrix

same result holds as F*

Extension to Separable Concave Program

Separable Concave Program over Unimodular Linear Space

$$F^{\mathrm{SC}}(w,c) \equiv \max\{\sum_{a \in A} f_a(\xi_a) \mid N\xi = \mathbf{0},\ \mathbf{0} \leq \xi \leq c\}$$

totally unimodular matrix

separable concave fn

P : "parallel" arc set, S: "series" arc set

F^{SC} is

(1) M-concave w.r.t. {c(a) | a∈P}

(2) L-concave w.r.t. {c(a) | a∈S}

Thanks for your attention!