Minimization Algorithms for M-convex Functions

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Outline of This Talk

- M-convex Function Minimization and Examples
- Greedy Algorithm
- Polynomial-time Scaling Algorithms
- Key Properties
  - Proximity Theorem
  - Minimizer Cut Property
  - Minimizer Cut Property with Scaling

M-convex Function Minimization

M-convex function (Murota 1996)

— desirable properties as "discrete convexity"

Minimize $f(x)$ subject to $x \in \mathbf{Z}^n$

— fundamental problem for M-convex fn

— includes various nonlinear integer programs

Examples of M-convex Function Minimization

- **Separable Convex Resource Allocation Problems**

Minimize $\sum_{i=1}^{n} f_i(x_i)$ subject to $\sum_{i=1}^{n} x_i = B,\ 0 \le x_i \le U_i,\ x \in \mathbf{Z}^n$

(each $f_i : \mathbf{Z} \to \mathbf{R}$ is convex)

- **A Class of Integer Quadratic Programs**

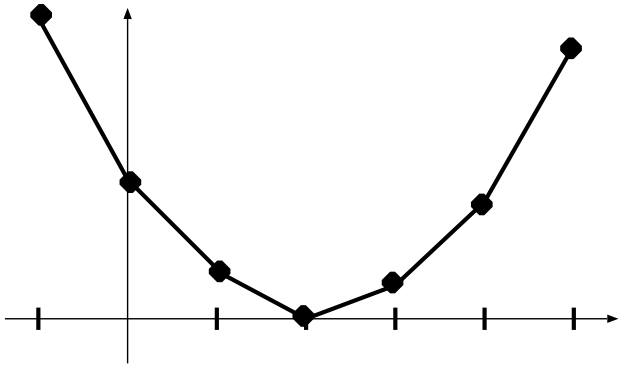
Minimize $x^{\top} A x$ subject to $\sum_{i=1}^{n} x_i = B,\ 0 \le x_i \le U_i,\ x \in \mathbf{Z}^n$

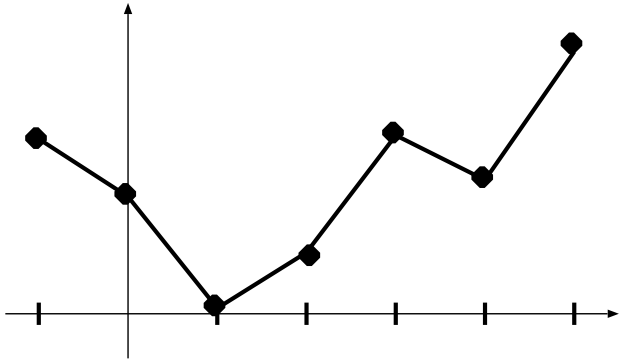
$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_2 & a_2 & a_3 & \cdots & a_n \\ a_3 & a_3 & a_3 & \cdots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_n & a_n & \cdots & a_n \end{bmatrix}, \quad a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_n$$

Greedy Algorithm for M-convex Function

- **Optimality Criteria** (Murota 1996)

f: M-convex $\Longrightarrow$ local opt = global opt





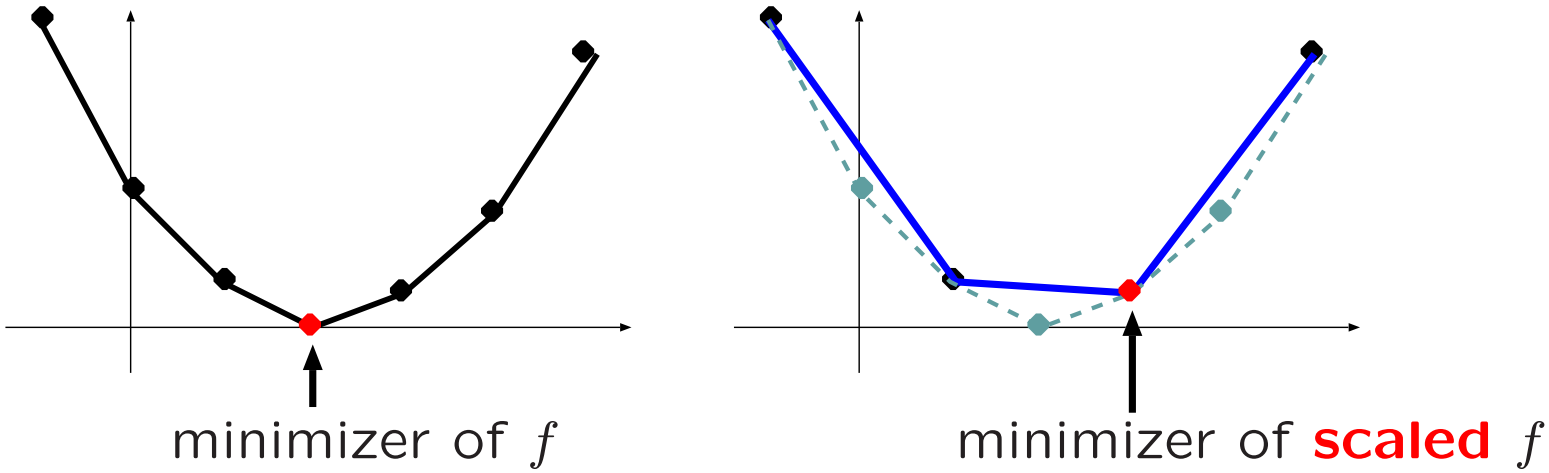
○ local opt = global opt

× local opt ≠ global opt

$\Longrightarrow$ greedy algorithm — exponential time

Key Properties for Polynomial Algorithms

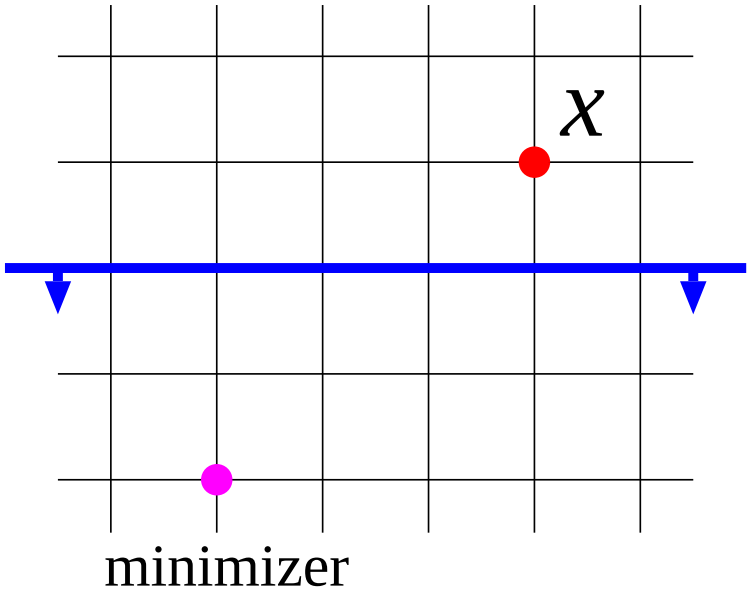
- **Proximity Theorem** (Moriguchi–Murota–Shioura 02)

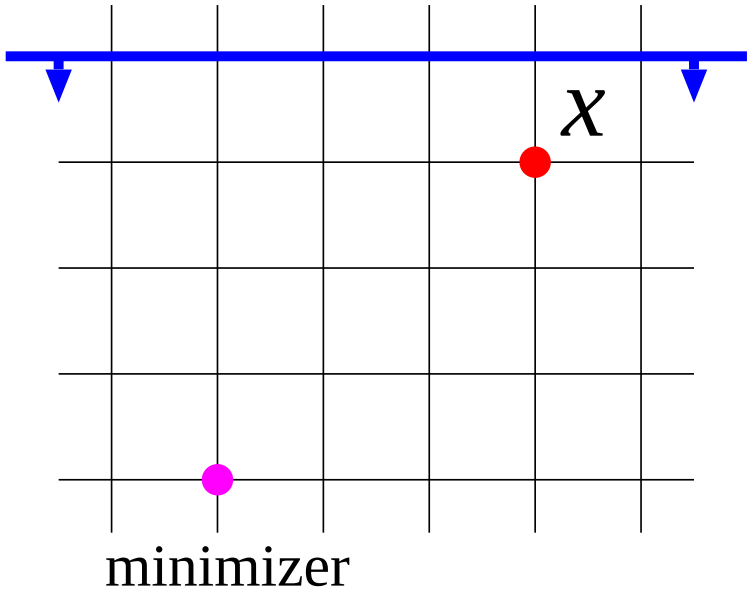


minimizer of $f \simeq$ minimizer of scaled f

Key Properties for Polynomial Algorithms (cont.)

- **Minimizer Cut Property** (Shioura 1998)
  — separation of a vector x from a minimizer





- **Minimizer Cut Property with Scaling** (Tamura 2002)
  — weak separation

Polynomial Algorithms

n: dimension, L: "size" of feasible region

- ○ ● ○ Shioura 98 — $O(n^4(\log L)^2)$
- ● ○ ○ Moriguchi–Murota–Shioura 02 — $O(n^3 \log L)$ for **restricted class**
- ○ ● ● Tamura 02 — $O(n^3 \log L)$
- ○ ○ ● Shioura 03 — $O(n^3 \log L)$
- ○ ● ○ Shioura 03 — $O((n^3 + n^2 \log L) \log_n L)$

Key Properties

- ● proximity theorem
- ● minimizer cut property
- ● minimizer cut property with scaling

Talk Plan

In the rest of my talk

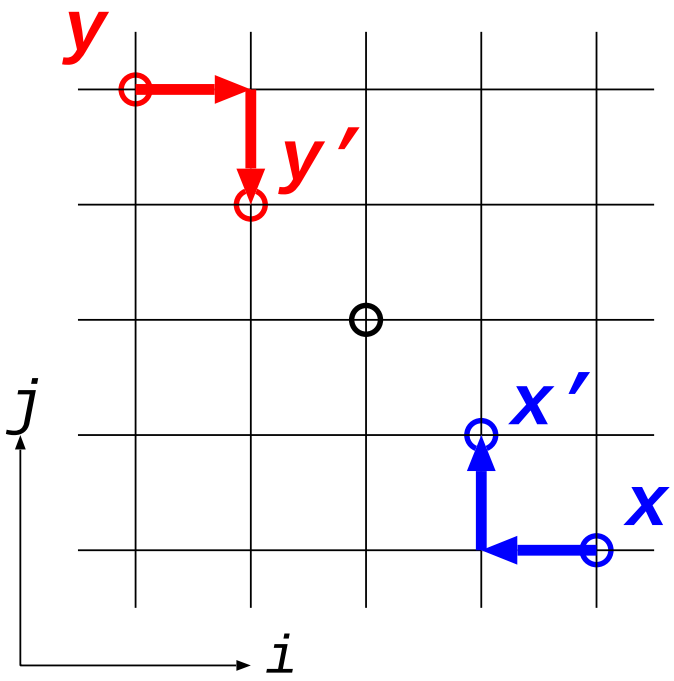
- Definition of M-convex Function
- Optimality Criteria: Local Opt = Global Opt
- Key Properties
  - Proximity Theorem
  - Minimizer Cut Property
  - Minimizer Cut Property with Scaling
- Scaling Algorithms
  - Moriguchi–Murota–Shioura 2002
  - Tamura 2002
  - Shioura 2003

Definition of M-convex Function

$f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ is **M-convex** $\overset{\text{def}}{\Longleftrightarrow}$

(M-EXC) $\forall x, y \in \operatorname{dom} f$, $\forall i \in \operatorname{supp}^+(x-y)$, $\exists j \in \operatorname{supp}^-(x-y)$:

$$f(x) + f(y) \geq f(x-\chi_i+\chi_j) + f(y+\chi_i-\chi_j)$$



$$\operatorname{dom} f = \{x \in \mathbf{Z}^n \mid f(x) < +\infty\}$$

$$\operatorname{supp}^+(x-y) = \{i \mid x_i > y_i\}$$

$$\operatorname{supp}^-(x-y) = \{i \mid x_i < y_i\}$$

χ_i: i-th unit vector

$$x \in \operatorname{dom} f \quad \Longrightarrow \quad \sum_{i=1}^{n} x_i = \text{const.}$$

Optimality Criteria

f: M-convex

neighborhood of $x \in \mathbf{Z}^n$ — $N(x) = \{x + \chi_i - \chi_j \mid i, j = 1, 2, \ldots, n\}$

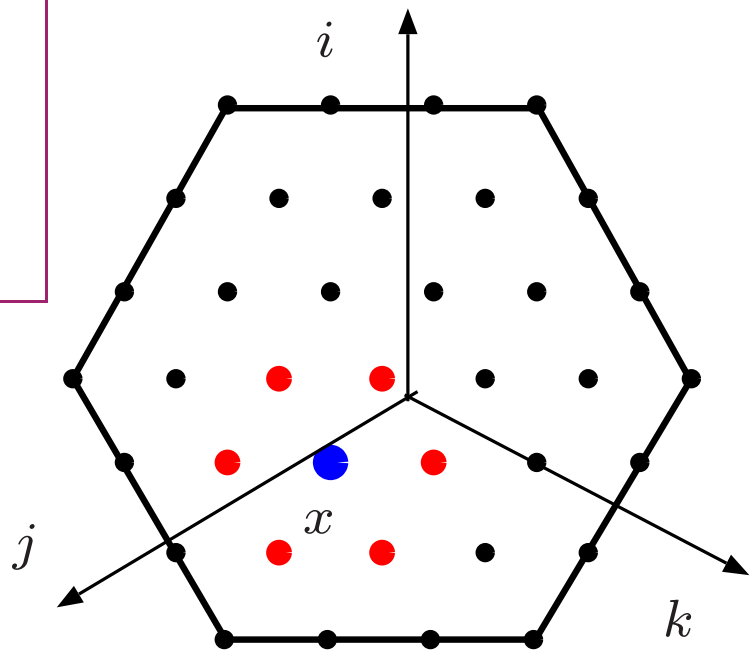
Property

x : minimizer in $N(x)$ $\Longrightarrow$ x: minimizer
(Local Opt) (Global Opt)

Greedy Algorithm

Find i, j minimizing $f(x + \chi_i - \chi_j)$
If $f(x + \chi_i - \chi_j) = f(x) \Longrightarrow x$: minimizer
Else, $x := x + \chi_i - \chi_j$

※ require exponential time



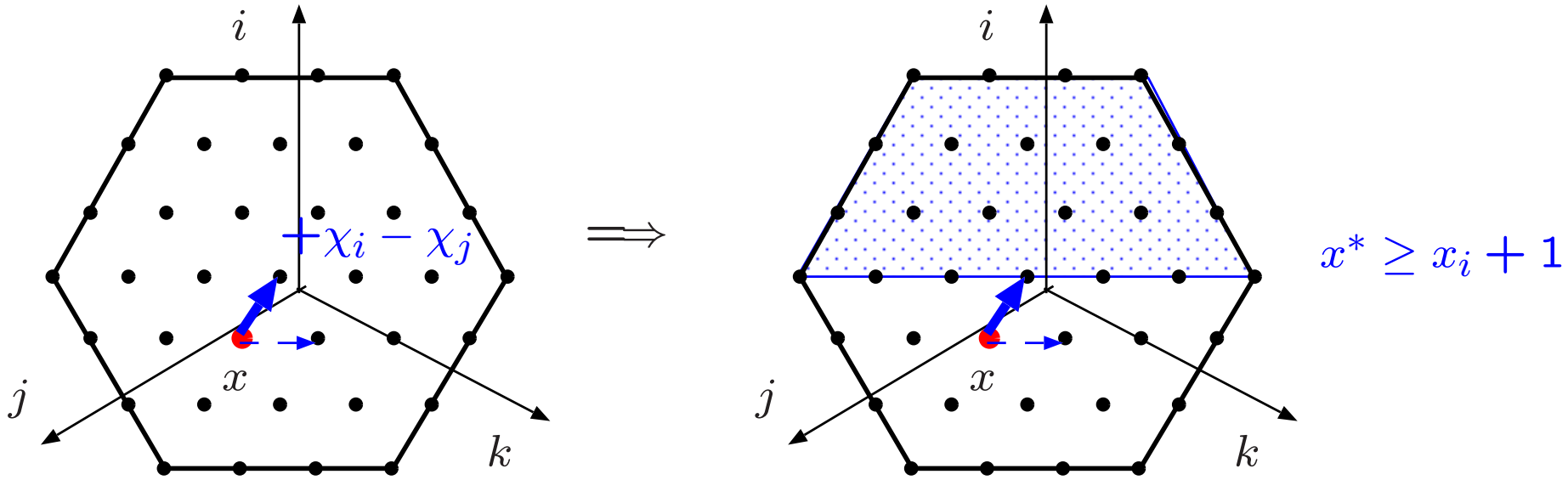
Minimizer Cut Property

- separation of a vector x from a minimizer

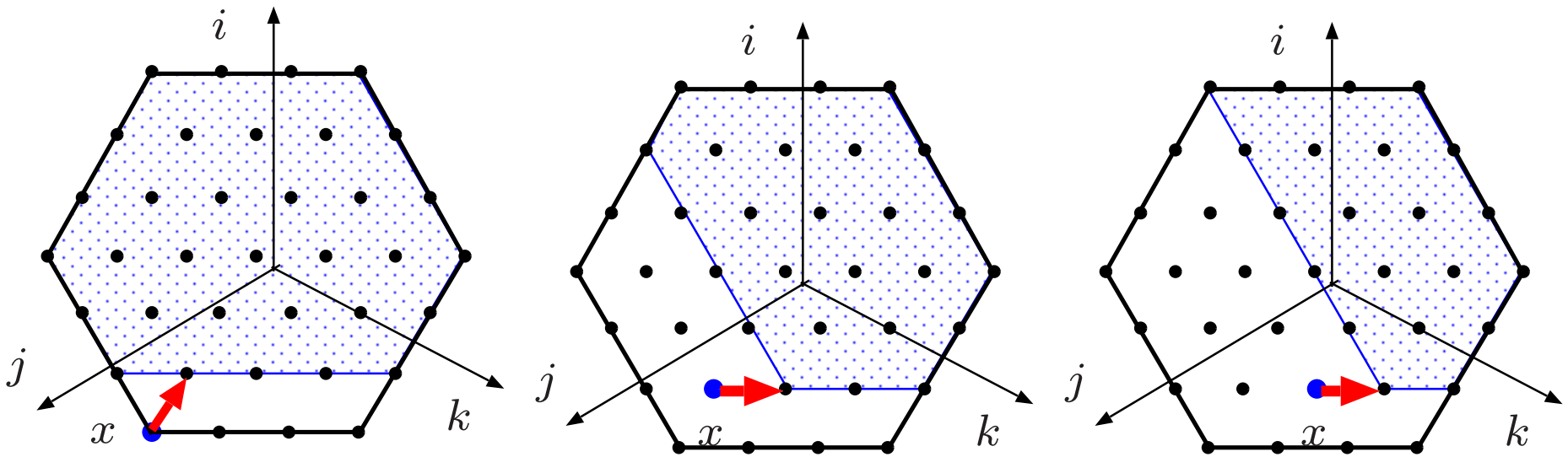
Minimizer Cut Property

$x \in \mathrm{dom}\, f, \quad j \in \{1, \ldots, n\}, \quad i : f(x + \chi_i - \chi_j) = \min_k f(x + \chi_k - \chi_j)$

$\Longrightarrow \exists\, x^*$: minimizer of f s.t. $x_i^* \geq x_i + 1$



Greedy Algorithm and Minimizer Cut Property



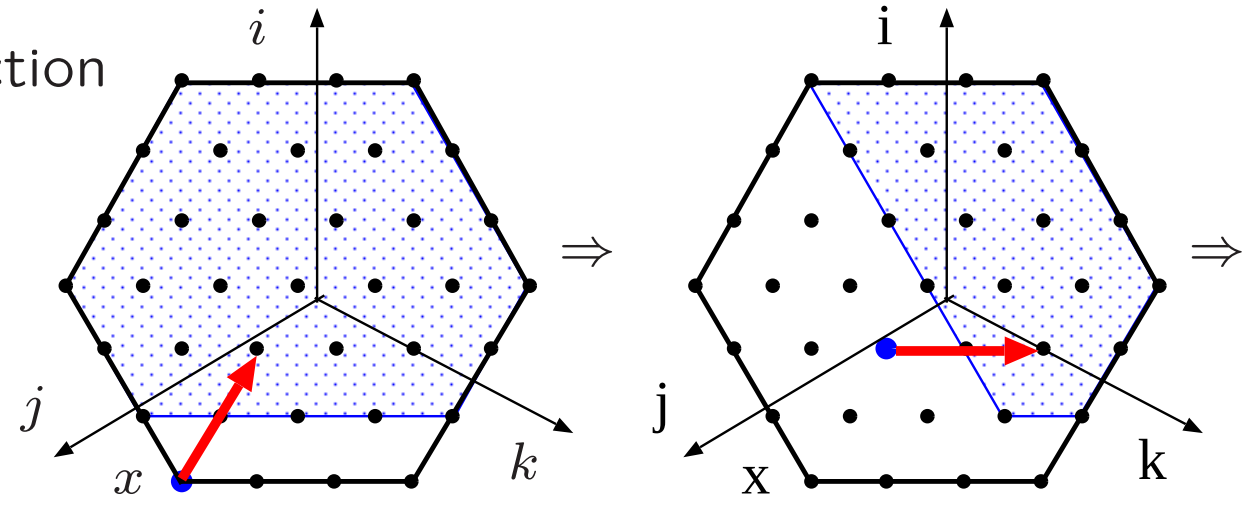
- choose greedy direction — $d = +\chi_i - \chi_j$ minimizing $f(x + \chi_i - \chi_j)$

  $\Longrightarrow$ reduce a set S containing a minimizer

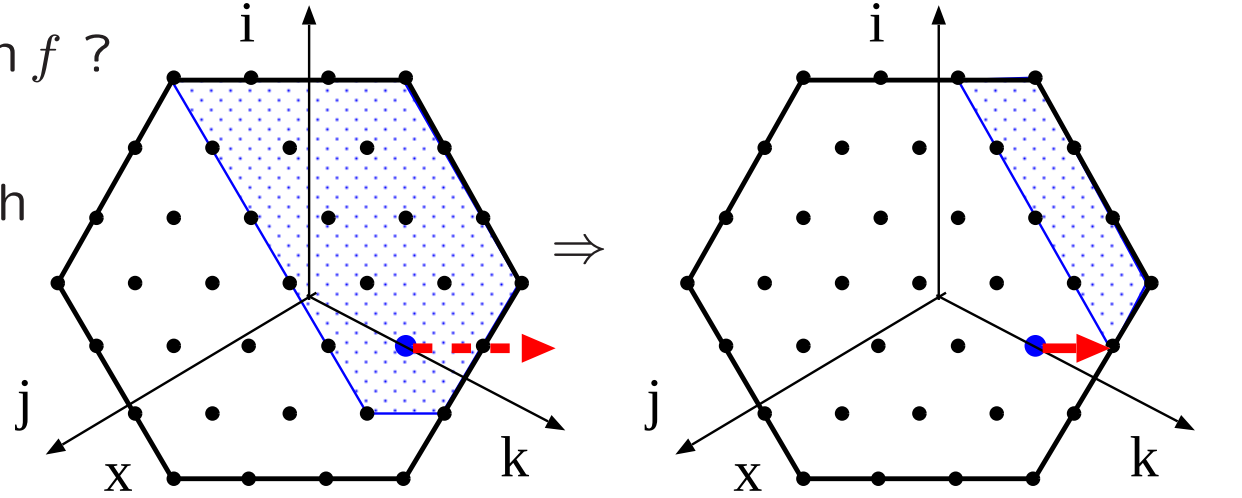
  $\Longrightarrow$ find a minimizer in $O(nL)$ iterations $\quad (L = \max_{x,y \in \mathrm{dom}\, f} ||x - y||_\infty)$

Scaling Algorithm Based on Minimizer Cut Prop.

- proceed along greedy direction
  by α steps
  $\Downarrow$
  reduce S substantially



- What if $x + \alpha(\chi_i - \chi_j) \notin \mathrm{dom}\, f$?
  $\Downarrow$
  Use maximum $\alpha'(< \alpha)$ with
  $x + \alpha'(\chi_i - \chi_j) \in \mathrm{dom}\, f$



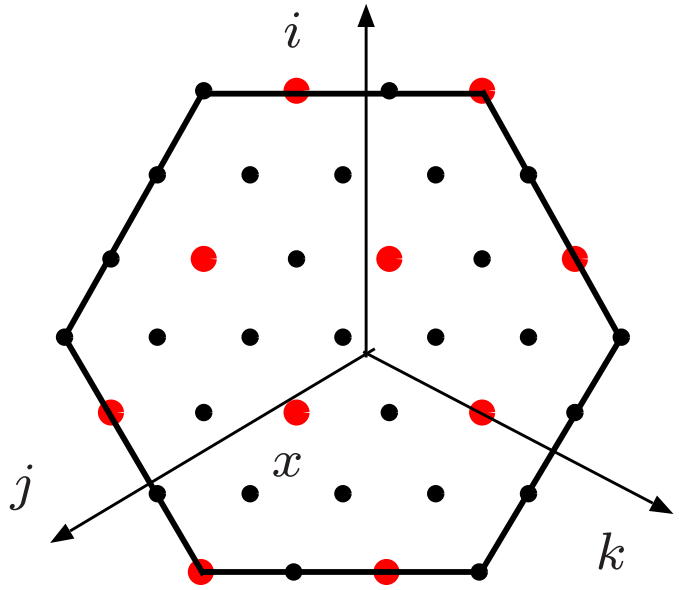
$\Longrightarrow$ $O((n^3 + n^2 \log L) \log_n L)$ time (Shioura 2003)

Proximity Theorem

Scaling of $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$

$$f_\alpha(y) = f(x_0 + \alpha y) \quad (y \in \mathbf{Z}^n)$$

$x_0 \in \mathbf{Z}^n$, $\alpha \in \mathbf{Z}_+$ (scaling parameter)



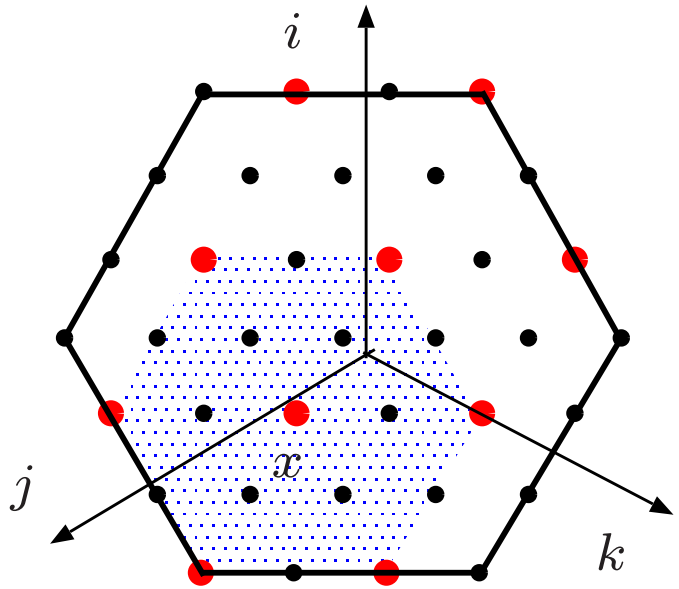
- minimizer of $f \simeq$ minimizer of f_α

Proximity Theorem

$f(x) \leq f(x + \alpha\chi_i - \alpha\chi_j)$ $(\forall i, j)$

$\Longrightarrow \exists x^*$: minimizer of f with

$$||x^* - x||_\infty \leq (n-1)(\alpha - 1)$$



Scaling Algorithm Based on Proximity Thm.

Maintain a set S containing a minimizer

Repeat until $\alpha < 1$
- Find a minimizer of f_α in S
- Reduce S by using Proximity Theorem
- $\alpha := \alpha/2$

$\forall \alpha > 0$, f_α is M-convex $\Longrightarrow$ minimization of f_α is easy

$\Longrightarrow$ $O(n^3 \log L)$ time (Moriguchi–Murota–Shioura 2002)

※ f_α is NOT M-convex in general

Minimizer Cut Property with Scaling

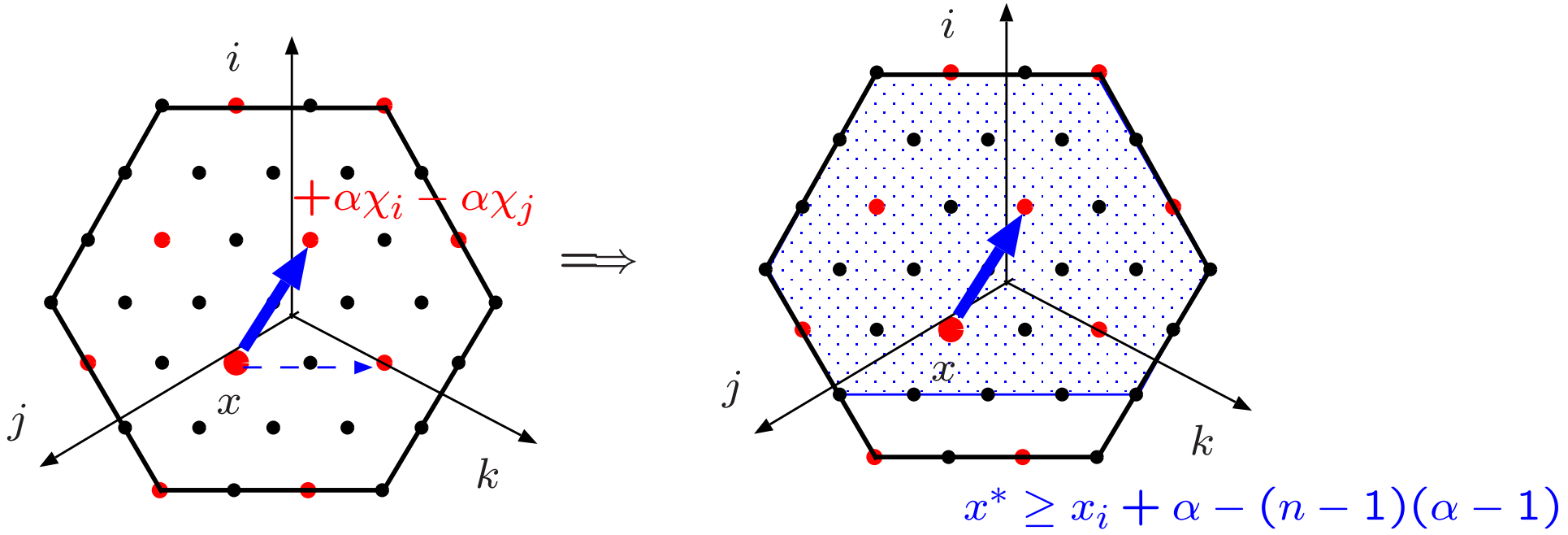
- weak separation of x from a minimizer

Minimizer Cut Property with Scaling

$x \in \mathrm{dom}\, f,\ j \in \{1, \ldots, n\},\ \alpha > 0$

$i : f(x + \alpha\chi_i - \alpha\chi_j) = \min_k f(x + \alpha\chi_k - \alpha\chi_j)$

$\Longrightarrow \exists\, x^*$: minimizer of f s.t. $\quad x_i^* \geq x_i + \alpha - (n-1)(\alpha - 1)$



Scaling Algorithms of Tamura 2002

- **Moriguchi et al. 2002:**
  - apply Greedy Algorithm to f_α to minimize f_α
  - $O(n^3 \log L)$ time if f_α is M-convex

- **Tamura 2002:**
  - apply Greedy Algorithm to f_α to reduce a set S

$\leftarrow$ Minimizer Cut Prop. with Scaling

$\Longrightarrow O(n^3 \log L)$ time for general M-convex functions

- **Shioura 2003:**
  - simplify the algorithm and analysis, $O(n^3 \log L)$ time

Summary and Future Work

Key Properties

- proximity theorem
- minimizer cut property
- minimizer cut property with scaling

- ○ ○ Moriguchi–Murota–Shioura 02 — $O(n^3 \log L)$ for **restricted class**
- ○ ● ● Tamura 02 — $O(n^3 \log L)$
- ○ ○ ● Shioura 03 — $O(n^3 \log L)$
- ○ ● ○ Shioura 03 — $O((n^3 + n^2 \log L) \log_n L)$

Future Work

- faster algorithms
- application to combinatorial optimization problems