Algorithms for M-convex Function Minimization

The Second Japanese–Sino Optimization Meeting
Kyoto, Japan, September 25–27, 2002

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Outline of this talk

- M-convex function minimization and its examples
- previous algorithms
- "local opt = global opt"
- minimizer cut property
- proximity theorem
- minimizer cut property with scaling

○ ● ○ ● M-convex Function Minimization ● ○ ● ○

M-convex function (Murota96)

— enjoys desirable properties as "discrete convexity"

M-convex function minimization

Minimize $f(x)$ subject to $x \in \mathbf{Z}^n$

— one of the fundamental problems concerning M-convex fn

— includes various nonlinear integer programs as special cases

Examples of M-convex function minimization

- **separable convex resource allocation problem**

Minimize $\sum_{i=1}^{n} f_i(x_i)$ subject to $\sum_{i=1}^{n} x_i = B,\ 0 \le x_i \le U_i,\ x \in \mathbf{Z}^n$

(each $f_i : \mathbf{Z} \to \mathbf{R}$ is convex)

- **certain class of integer quadratic program**

Minimize $x^{\mathsf{T}} A x$ subject to $\sum_{i=1}^{n} x_i = B,\ 0 \le x_i \le U_i,\ x \in \mathbf{Z}^n$

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_2 & a_2 & a_3 & \cdots & a_n \\ a_3 & a_3 & a_3 & \cdots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_n & a_n & \cdots & a_n \end{bmatrix},\ a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_n$$

○ ● ○ ● Previous Algorithms ● ○ ● ○

n: dimension of vectors, L: "size" of feasible region

- greedy algorithms — exponential time by (steepest) descent methods
- Shioura98 — $O(n^4(\log L)^2)$, first polynomial-time algorithm
- Moriguchi et al.02 — scaling approach, $O(n^3 \log L)$ for restricted class
- Tamura02 — $O(n^3 \log L)$, polish up the approach of Moriguchi et al.02
- Shioura02a — $O(n^3 \log L)$
- Shioura02b — $O((n^3 + n^2 \log L) \log_n L)$
  greedy algorithm + scaling

Useful properties of M-convex functions

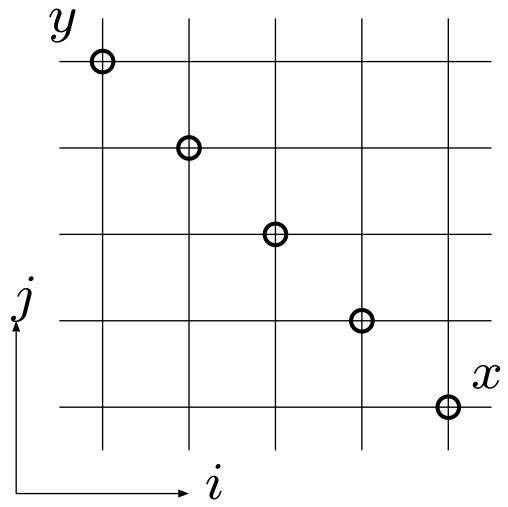
- local opt = global opt
- minimizer cut property
- proximity theorem
- minimizer cut property with scaling

Definition of M-convex function

$f : \mathbf{Z}^n \to \mathbf{Z} \cup \{+\infty\}$ is **M-convex** $\overset{\text{def}}{\Longleftrightarrow}$ (M-EXC)

(M-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y)$:

$$f(x) + f(y) \geq f(x-\chi_i+\chi_j) + f(y+\chi_i-\chi_j)$$



$\mathrm{dom}\, f = \{x \in \mathbf{Z}^n \mid f(x) < +\infty\}$
$\mathrm{supp}^+(x-y) = \{i \mid x_i > y_i\}$
$\mathrm{supp}^-(x-y) = \{i \mid x_i < y_i\}$
χ_i: i-th unit vector

Proposition

f: M-convex $\Longrightarrow \exists B \in \mathbf{Z}$ such that $\sum_{i=1}^{n} x_i = B$ $(\forall x \in \mathrm{dom}\, f)$

“Local Opt = Global Opt” Property

f: M-convex, $N = \{1, 2, \ldots, n\}$, $\arg\min f$: set of minimizers

neighborhood of $x \in \mathbf{Z}^n$ — $\{x + \chi_i - \chi_j \mid i, j = 1, 2, \ldots, n\}$

“Local opt = global opt” property [Murota96]

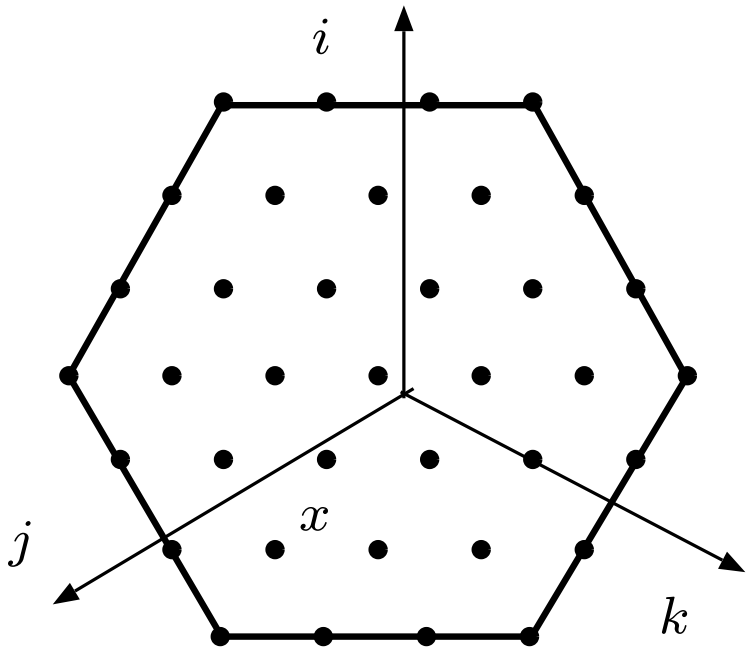
x : minimizer in the neighborhood

$\Longrightarrow x \in \arg\min f$

Descent Method

Check if $x \in \arg\min f$.
If not, find $i, j \in N$ with
$f(x + \chi_i - \chi_j) < f(x)$.
Put $x := x + \chi_i - \chi_j$.

※ require exponential time



Minimizer Cut Property

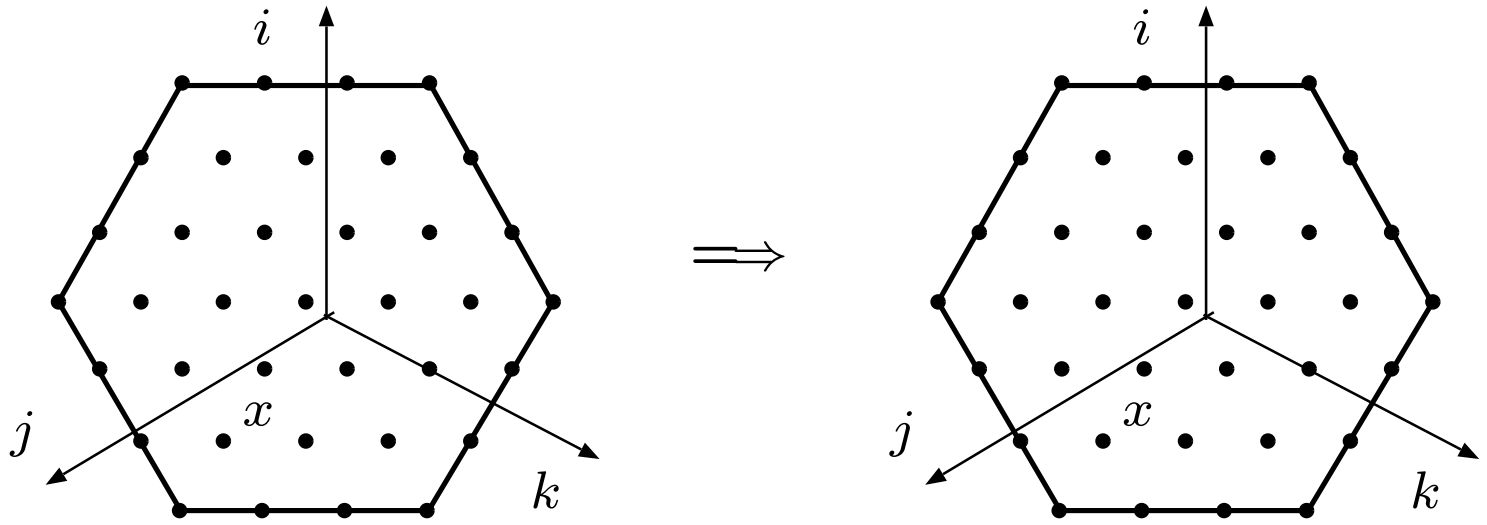
- speedup of descent method

Minimizer cut property (Shioura98)

$x \in \mathrm{dom}\, f,\ j \in N, \quad i \in N : f(x + \chi_i - \chi_j) = \min_{k \in N} f(x + \chi_k - \chi_j)$

$$\Longrightarrow \exists x^* \in \arg\min f \text{ with } x^*_i \geq \begin{cases} x_i + 1 & (i \neq j), \\ x_i & (i = j). \end{cases}$$

$\Longrightarrow$ can cut feasible reagion S in $O(n)$ time



Steepest Descent Method

Descent Method + minimizer cut property $\Rightarrow$ Steepest Descent Method

Steepest Descent Method

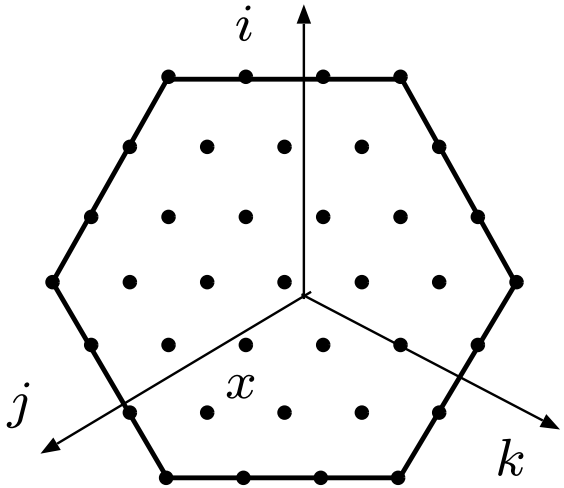
$S = \mathrm{dom}\, f$, x: initial vector.
Check if $x \in \arg\min f$.
If not, choose $j \in N$ appropriately.
Find $i \in N$ with

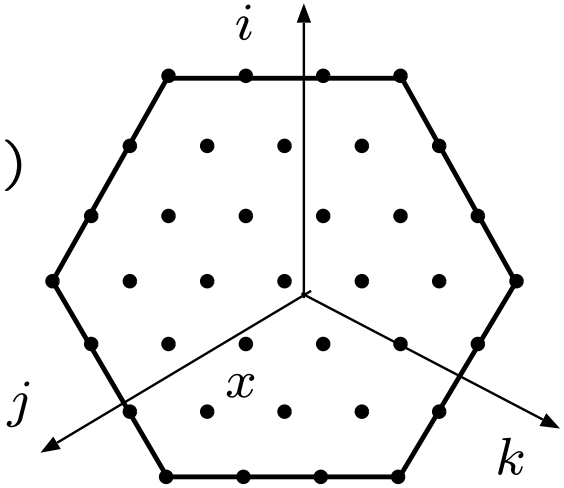
$$f(x + \chi_i - \chi_j) = \min_{k \in N} f(x + \chi_k - \chi_j).$$

Put $x := x + \chi_i - \chi_j$,

$$S := \begin{cases} S \cap \{y \mid y_i \geq x_i + 1\} & (i \neq j), \\ S \cap \{y \mid y_i \geq x_i\} & (i = j). \end{cases}$$

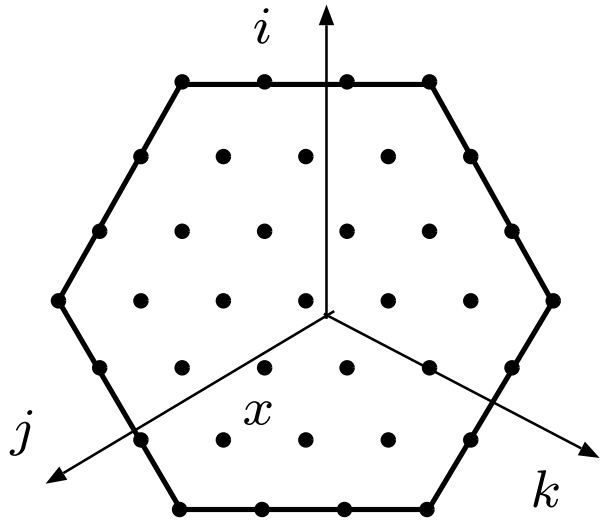
$\Longrightarrow$ $O(n^2 L)$-time algorithm ($L = \max_{x,y \in \mathrm{dom}\, f} ||x - y||_\infty$)





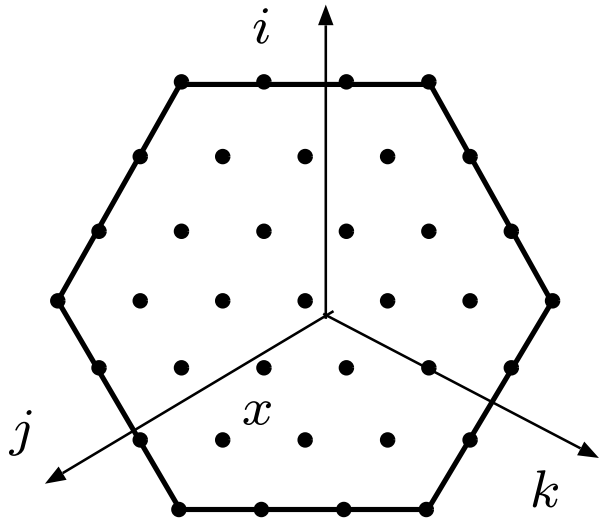
Domain Reduction Method (Shioura98)

minimizer cut property $\simeq$ separation of a vector x from a minimizer



bad choice of x

$\Rightarrow$ S almost unchanged



good choice of x

$\Rightarrow$ S decreased substantially

Domain Reduction Method (Shioura98)

find x in the "center" of $S \rightarrow$ cut feasible region S

$\Longrightarrow$ $O(n^4(\log L)^2)$-time algorithm $\quad (L = \max_{x,y \in \mathrm{dom}\, f} ||x - y||_\infty)$

Steepest Descent with Scaling (Shioura02)

Steepest Descent + scaling technique $\Longrightarrow$ faster algorithm

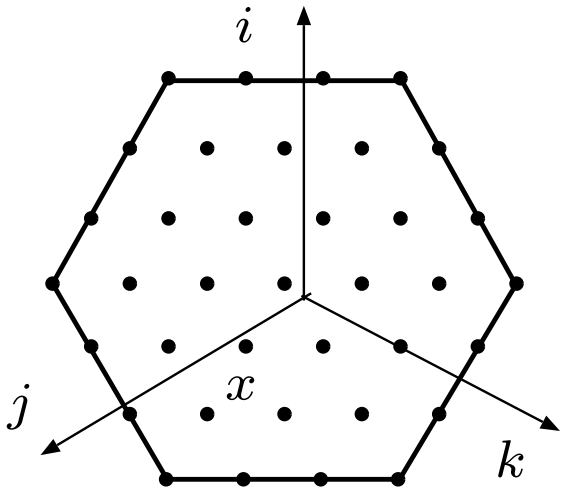
each scaling phase — steepest descent with step length $\alpha > 0$

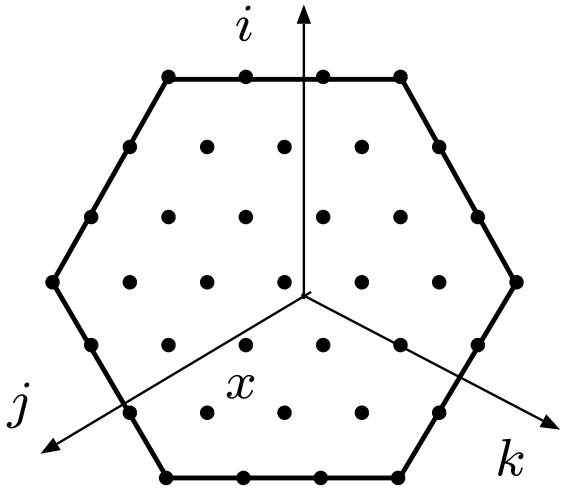
$S \subseteq \mathbf{Z}^n$: subset of dom f
containing a minimizer, $x \in S$.
Choose $j \in N$ appropriately.
Find $i \in N$ with

$$f(x + \chi_i - \chi_j) = \min_{k \in N} f(x + \chi_k - \chi_j).$$

Put $x := x + \alpha\chi_i - \alpha\chi_j$,

$$S := \begin{cases} S \cap \{y \mid y_i \geq x_i + 1\} & (i \neq j), \\ S \cap \{y \mid y_i \geq x_i\} & (i = j). \end{cases}$$





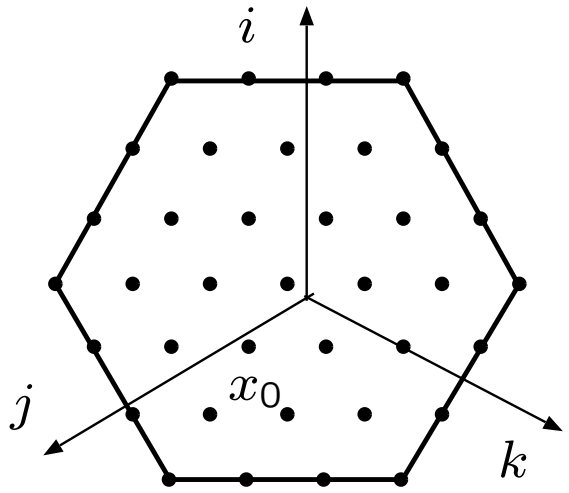
- naive implementation does not work
- + some modification
  $\Longrightarrow O((n^3 + n^2 \log \frac{L}{n}) \log_n \frac{L}{n})$ algorithm

Proximity Theorem and Scaling Approach

scaling of an M-convex function $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$

$x_0 \in \mathbf{Z}^n$, $\alpha \in \mathbf{Z}_+$ (scaling parameter)

$$f_\alpha(y) = f(x_0 + \alpha y) \quad (y \in \mathbf{Z}^n)$$



Proximity theorem (Moriguchi et al.02)

$f(x) \le f(x + \alpha\chi_i - \alpha\chi_j)$ $(\forall i, j \in N)$

$\Longrightarrow \exists\, x^* \in \arg\min f$ with $|x_i^* - x_i| < (n-1)(\alpha - 1)$ $(\forall i \in N)$

if $\forall \alpha > 0$, f_α is M-convex $\Longrightarrow$ minimization of f_α is easy

$\Longrightarrow$ $O(n^3 \log L)$ time (Moriguchi et al.02)

Minimizer Cut Property with Scaling

※ f_α is NOT M-convex in general $\Longrightarrow$ minimization of f_α is difficult

generalization of $\left\{ \begin{array}{c} \text{minimizer cut property} \\ \text{proximity theorem} \end{array} \right\} \Rightarrow$ more useful property for f_α

Minimizer cut property with scaling (Tamura02)

$x \in \operatorname{dom} f$, $j \in N$, $\alpha > 0$

$i \in N : f(x + \alpha\chi_i - \alpha\chi_j) = \min_{k \in N} f(x + \alpha\chi_k - \alpha\chi_j)$

$\Longrightarrow \exists\, x^* \in \arg\min f$ with

$$x_i^* > \begin{cases} x_i + \alpha - (n-1)(\alpha - 1) & (i \neq j), \\ x_i - (n-1)(\alpha - 1) & (i = j). \end{cases}$$

※ coincides with minimizer cut property when $\alpha = 1$

apply steepest descent alg. to f_α

$\Longrightarrow$ reduce the feasible region $S \subseteq \operatorname{dom} f$

$\Longrightarrow$ $O(n^3 \log \frac{L}{n})$-time algorithm (Tamura02, Shioura02a)