# Quasi M-convex and L-convex Functions: Quasi-Convexity in Discrete Optimization

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## ○ ● ○ ● Outline ● ○ ● ○

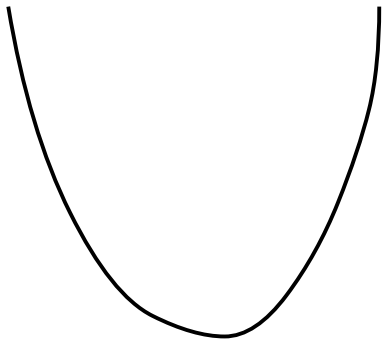
- Convex and Quasi-Convex Functions in Continuous Optimization
- M-Convex Function — Convexity in Discrete Optimization
- Minimization Algorithms for M-convex Fn
- Definition of Quasi M-Convex Fn
- Various Properties on Quasi M-Convex Fn

## ○ • Convexity in Continuous Optimization • ○

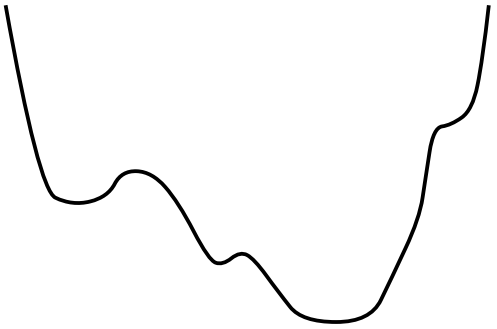
**Continuous Optimization:** $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$

| **Minimize** | $f(x)$ | **subject to** | $x \in \mathbf{R}^n$ |
|---|---|---|---|

**efficiently solvable** $\simeq$ $f$**: convex**



**convex**



**nonconvex**

$f$**: convex** $\Longrightarrow$ **local opt = global opt**

$\Longrightarrow$ **Descent Method works!**
**(Greedy Algorithm)**

## ○ ● ○ ● Quasi-Convex Function ● ○ ● ○

**(semistrict) Quasi-Convexity**

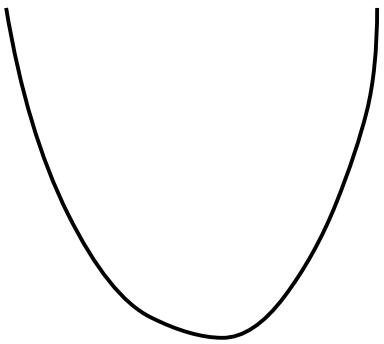
$\Longrightarrow$ **"local opt = global opt"**

$\forall x, y \in \mathbf{R}^n,\ \alpha \in [0,1]$**:**

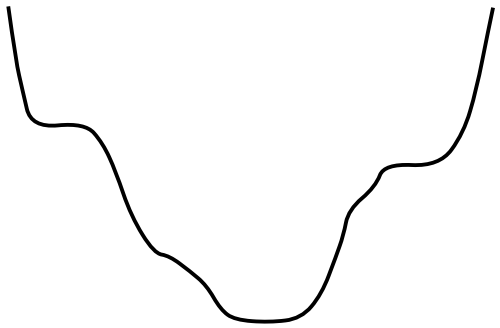
$f$**: convex** $\overset{\text{def}}{\Longleftrightarrow}\ \alpha f(x) + (1-\alpha) f(y) \geq f(\alpha x + (1-\alpha) y)$

$f$**: quasi-convex** $\overset{\text{def}}{\Longleftrightarrow}\ \max\{f(x), f(y)\} \geq f(\alpha x + (1-\alpha) y)$

**convex**



**quasi-convex**



- $f$: **quasi-convex** $\Longleftrightarrow$ **Level Sets: convex**

  $\mathsf{L}(f, \alpha) = \{x \in \mathbf{R}^n \mid f(x) \leq \alpha\}$

- **Descent method works**

  **for semistrictly quasi-conv fn**

## • ○ • Convexity in Discrete Optimization ○ • ○

**Discrete Optimization:** $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$

| **Minimize** $f(x)$ **subject to** $x \in \mathbf{Z}^n$ |
|---|

**What structure needed for Greedy Algorithm?**

**$\Rightarrow$ Greedy Alg $\simeq$ Descent Method**

**$\Rightarrow$ Discrete Convexity (Miller71, Favati–Tardella90)**

**M-Convex fn (Murota96)**

- **local opt = global opt**
- **extend to convex fn**
- **duality theorems**

**$\Longrightarrow$ greedy algorithm works**

**"Quasi M-Convexity" suffices?**

- **Definitions**
- **Properties, Algorithms**

## ○ ● ○ ● Definition of M-Convex Fn ● ○ ● ○

$f : \mathbf{Z}^n \to \mathbf{Z} \cup \{+\infty\}$ — **M-Convex** $\overset{\text{def}}{\Longleftrightarrow}$ **(M-EXC)**

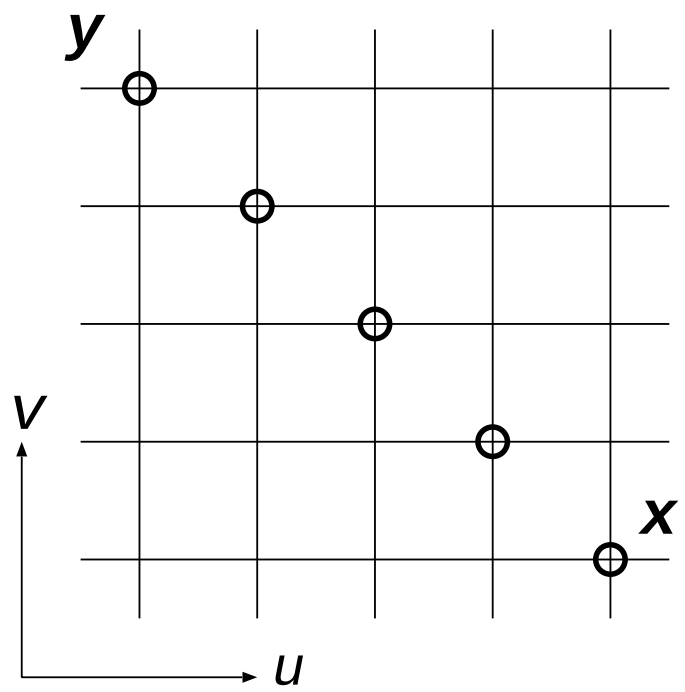
**(M-EXC)**

$\forall x, y \in \mathrm{dom}\, f,\ \forall i \in \mathrm{supp}^+(x-y),\ \exists j \in \mathrm{supp}^-(x-y)$:

$$f(x) + f(y) \geq f(x-\chi_i+\chi_j) + f(y+\chi_i-\chi_j)$$

$\mathrm{dom}\, f = \{x \in \mathbf{Z}^n \mid f(x) < +\infty\}$,
$\mathrm{supp}^+(x-y) = \{i \mid x_i > y_i\}$,
$\mathrm{supp}^-(x-y) = \{i \mid x_i < y_i\}$,

$$\chi_i = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$



## ○ • Minimization Algorithms for M-conv Fn • ○

- **Descent Method** — **exponential**
- **Steepest Descent** — **pseudopolynomial**
- **Domain Reduction** — **(weakly) polynomial**

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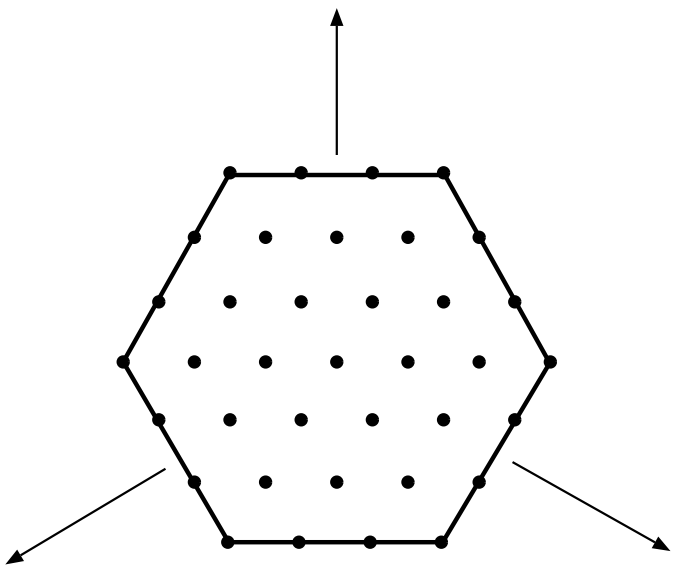
## (1) Descent Method

**Theorem 1 (Murota 96)**

$f(x) \leq f(x-\chi_i+\chi_j)\ (\forall i,j) \Longleftrightarrow f(x) \leq f(y)\ (\forall y \in \mathbf{Z}^n)$

**"local opt = global opt"**

$\Longrightarrow$ **descent method**



**S1:** $f(x) = \min_{i,j} f(x - \chi_i + \chi_j) \Longrightarrow x$: global opt.

**S2:** Find $i, j \in \{1, \cdots, n\}$ s.t. $f(x - \chi_i + \chi_j) < f(x)$.

**S3:** $x := x - \chi_i + \chi_j$. Go to S1.

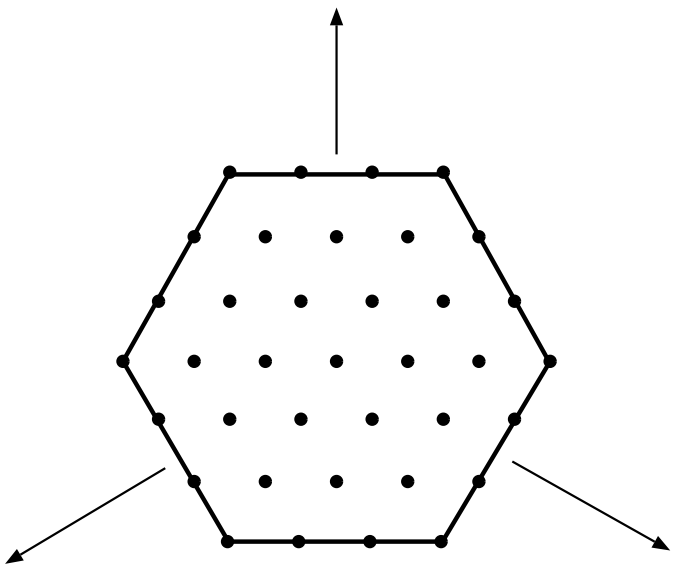
## (2) Steepest Descent Method

**S2′:** Find $i, j \in \{1, \cdots, n\}$ s.t.
$$f(x - \chi_i + \chi_j) = \min_{h,k} f(x - \chi_h + \chi_k)$$

**Theorem 2 (Shioura 99)** $x \in \mathrm{dom}\, f$, **not optimal,**
$f(x - \chi_i + \chi_j) = \min_{h,k} f(x - \chi_h + \chi_k)$
$\Longrightarrow \exists x^* \in \arg\min f$**:** $x^*_i \leq x_i - 1$, $x^*_j \geq x_j + 1$.

$\Rightarrow$ $O(nL)$ **iter.**

$$L = \max_{x,y \in \mathrm{dom}\, f} ||x - y||_\infty$$



## (3) Domain Reduction Mtd

**Theorem 2**
$\Longrightarrow$ **separation of any vector and opt. sol. possible!**
$\Longrightarrow$ **# of iter.** $= O(n^2 \log L)$**, time** $= O(n^4 (\log L)^2)$

## ○ ● ○ ● Def of Quasi M-Convexity ● ○ ● ○

$f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$

$\Delta f(x; j, i) = f(x - \chi_i + \chi_j) - f(x)$: **directional diff.**

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$\forall x, y \in \mathrm{dom}\, f \ (x \neq y)$,

$\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y)$:

| | |
|---|---|
| **(M-EXC)** | $\Delta f(x; j, i) + \Delta f(y; i, j) \leq 0$ |
| **(SSQM)** | $\Delta f(x; j, i) \geq 0 \Longrightarrow \Delta f(y; i, j) \leq 0$, **and** $\Delta f(y; i, j) \geq 0 \Longrightarrow \Delta f(x; j, i) \leq 0$ |
| **(QM)** | $\Delta f(x; j, i) \leq 0$ **or** $\Delta f(y; i, j) \leq 0$ |

**(QM)** — **Quasi M-Convex**

**(SSQM)** — **Semistrictly Quasi M-Convex**

**(M-EXC) $\Longrightarrow$ (SSQM) $\Longrightarrow$ (QM)**

## ○ ● ○ ● Properties of Quasi M-Convexity ● ○ ● ○

○ $f$ : **(SSQM)**
$\Longrightarrow$ **statements of Theorems 1,2 hold**
$\Longrightarrow$ **Minimization algorithms also work for quasi M-convex functions!**

○ **Quasi M-convex Fn and Level Sets**

**Level Sets:** $\mathsf{L}(f,\alpha) = \{x \in \mathbf{R}^n \mid f(x) \le \alpha\}$

**(Q-EXC$_{\mathsf{W}}$) $\cdots$ Quasi M-Convexity for** $S \subseteq \mathbf{Z}^n$

$\forall x, y \in S$ $(x \neq y)$, $\exists i \in \mathsf{supp}^+(x-y)$, $\exists j \in \mathsf{supp}^-(x-y)$:
$x - \chi_i + \chi_j \in S$ **or** $y + \chi_i - \chi_j \in S$

**(QM$_{\mathsf{W}}$) — weaker version of (QM)**

**Theorem 3**
$f$: **(QM$_{\mathsf{W}}$)** $\Longleftrightarrow$ $\forall \alpha$, $\mathsf{L}(f,\alpha)$: **(Q-EXC$_{\mathsf{W}}$)**