Substitutes and Complements in Network Flows Viewed as Discrete Convexity

Kazuo MUROTA[1] and Akiyoshi SHIOURA[2]

October, 2002; revised May, 2004; April, 2005

Abstract

We study combinatorial properties of the optimal value function of the network flow problem. It is shown by Gale–Politof (1981) that the optimal value function has submodularity and supermodularity w.r.t. problem parameters such as weights and capacities. In this paper we shed a new light on this result from the viewpoint of discrete convex analysis to point out that the submodularity and supermodularity are naturally implied by discrete convexity, called M-convexity and L-convexity, of the optimal value function.

Keywords: network flow, submodularity, discrete convexity, combinatorial optimization

[1]Graduate School of Information Science and Technology, University of Tokyo, Tokyo 113-8656, Japan; and PRESTO, JST, `murota@mist.i.u-tokyo.ac.jp`.

[2]Graduate School of Information Sciences, Tohoku University, Sendai 980-8579, Japan, `shioura@dais.is.tohoku.ac.jp`.

1 Introduction

In this paper, we study combinatorial properties of the optimal value function of the network flow problem. It is shown by Gale–Politof [4] that the optimal value function has submodularity and supermodularity w.r.t. problem parameters such as weights and capacities. On the other hand, it is well known in parametric linear programming that the optimal value function has convexity and concavity w.r.t. problem parameters. The main aim of this paper is to shed a new light on these results from the viewpoint of discrete convex analysis introduced by Murota [12, 13], and show that submodularity/supermodularity and convexity/concavity are naturally implied by discrete convexity called M-convexity and L-convexity of the optimal value function. From the viewpoint of mathematical economics, our result reveals that the optimal value function of network flow problem has nicer (stronger) combinatorial properties such as the gross substitutes property than submodularity and supermodularity.

1.1 Substitutes and Complements in Network Flows

Let $G = (V, A)$ be a directed graph with vertex set V and arc set A, and $N \in \{0, +1, -1\}^{V \times A}$ be the vertex-arc incidence matrix of G. A flow $\xi = (\xi(a) \mid a \in A)$ is called a *circulation* if it satisfies the conservation constraint $N\xi = \mathbf{0}$, which can be written as

$$\sum\{\xi(a) \mid a \text{ leaves } v\} - \sum\{\xi(a) \mid a \text{ enters } v\} = 0 \qquad (v \in V).$$

For each arc $a \in A$, we are given an upper bound $c(a)$ and a lower bound $d(a)$ for flow in a and a weight $w(a)$ per unit flow. The *maximum weight circulation problem* is to find a circulation ξ that maximizes the total weight $\sum_{a\in A} w(a)\xi(a)$ subject to the capacity (feasibility) constraint:

$$d(a) \le \xi(a) \le c(a) \qquad (a \in A).$$

We denote by F^{NF} the maximum weight of a feasible circulation, i.e.,

$$F^{\mathrm{NF}} = \max\{w^{\mathrm{T}}\xi \mid N\xi = \mathbf{0},\ d \le \xi \le c\}. \tag{1.1}$$

Our concern here is how the weight F^{NF} depends on the problem parameters (w, c, d). Namely, we are interested in the function $F^{\mathrm{NF}} = F^{\mathrm{NF}}(w, c, d)$ in $w \in \mathbf{R}^A$ and $c, d \in \mathbf{R}^A$. We also consider the case where parameters are restricted to be integral, and denote by $F^{\mathrm{NF}}_{\mathbf{Z}}$ the function F^{NF} restricted to integer parameters $(w, c, d) \in \mathbf{Z}^A \times \mathbf{Z}^A \times \mathbf{Z}^A$.

We first look at submodularity and supermodularity. Two arcs are said to be *"parallel"* if every (undirected) simple cycle containing both of them orients them in the opposite direction, and *"series"* if every (undirected) simple cycle containing both of them orients them in the same direction. A set of arcs is said to be *"parallel"* if it consists of pairwise "parallel" arcs, and *"series"* if it consists of pairwise "series" arcs. With notations $w_P = (w(a) \mid a \in P)$, $c_P = (c(a) \mid a \in P)$, $d_P = (d(a) \mid a \in P)$, $w_S = (w(a) \mid a \in S)$, $c_S = (c(a) \mid a \in S)$, and $d_S = (d(a) \mid a \in S)$, the following statements hold true.

Theorem 1.1 (Gale–Politof [4]). *Let P be a "parallel" arc set and S a "series" arc set.*
(i) $F^{\rm NF}$ *is submodular in w_P, in c_P, and in d_P.*
(ii) $F^{\rm NF}$ *is supermodular in w_S, in c_S, and in d_S.*

See [5, 6, 7, 8, 21] for extensions of this result.

As for convexity and concavity, the following is a well-known fact in parametric linear programming.

Proposition 1.2. $F^{\rm NF}$ *is convex in w and concave in c and in d.*

Combining Theorem 1.1 and Proposition 1.2 yields that

$$\left.\begin{array}{llclll} F^{\rm NF} & \text{is} & \text{submodular} & \text{and} & \text{convex} & \text{in } w_P, \\ F^{\rm NF} & \text{is} & \text{submodular} & \text{and} & \text{concave} & \text{in } c_P \text{ and in } d_P, \\ F^{\rm NF} & \text{is} & \text{supermodular} & \text{and} & \text{convex} & \text{in } w_S, \\ F^{\rm NF} & \text{is} & \text{supermodular} & \text{and} & \text{concave} & \text{in } c_S \text{ and in } d_S. \end{array}\right\} \qquad (1.2)$$

Thus all combinations of submodularity/supermodularity and convexity/concavity arise in our network flow problem. Although submodularity and convexity are mutually independent properties in general, the combinations of submodularity/supermodularity and convexity/concavity in (1.2) are not accidental phenomena but logical consequences that can be explained in terms of M-convexity and L-convexity.

The concepts of M-convex and L-convex functions are introduced by Murota [12, 13], aiming to identify a well-behaved structure in nonlinear combinatorial optimization. These concepts were originally defined for functions over the integer lattice; subsequently, their variants called M$^\natural$-convexity and L$^\natural$-convexity were introduced by Murota–Shioura [15] and by Fujishige–Murota [2], respectively. M-/M$^\natural$-convex and L-/L$^\natural$-convex functions enjoy a number of nice properties that are expected of "discrete convex functions" [14]. In general, L$^\natural$-convexity implies submodularity by definition, whereas M$^\natural$-convexity implies supermodularity [17]. Accordingly, L$^\natural$-concavity implies supermodularity and M$^\natural$-concavity submodularity. Recently, Murota–Shioura [16, 18] extended these concepts to convex functions defined over the real space, aiming at clarifying a well-behaved structure in nonlinear combinatorial optimization problems in continuous variables. It is shown that most of the previous combinatorial results extend to M-/M$^\natural$-convex and L-/L$^\natural$-convex functions over the real space.

In this paper, we show that the function $F^{\rm NF}$ (and $F^{\rm NF}_{\mathbf{Z}}$) defined by (1.1) is endowed with M$^\natural$-convexity and L$^\natural$-convexity as follows, where the definitions of M$^\natural$-convexity and L$^\natural$-convexity are given in Section 2.

Theorem 1.3. *Let P be a "parallel" arc set and S a "series" arc set.*
(i) $F^{\rm NF}$ *is L$^\natural$-convex in $w_P \in \mathbf{R}^P$ and M$^\natural$-convex in $w_S \in \mathbf{R}^S$.*
(ii) $F^{\rm NF}$ *is M$^\natural$-concave in $c_P \in \mathbf{R}^P$ and L$^\natural$-concave in $c_S \in \mathbf{R}^S$.*
(iii) $F^{\rm NF}$ *is M$^\natural$-concave in $d_P \in \mathbf{R}^P$ and L$^\natural$-concave in $d_S \in \mathbf{R}^S$.*

Theorem 1.4. *Let P be a "parallel" arc set and S a "series" arc set.*
(i) $F^{\rm NF}_{\mathbf{Z}}$ *is L$^\natural$-convex in $w_P \in \mathbf{Z}^P$ and M$^\natural$-convex in $w_S \in \mathbf{Z}^S$.*
(ii) $F^{\rm NF}_{\mathbf{Z}}$ *is M$^\natural$-concave in $c_P \in \mathbf{Z}^P$ and L$^\natural$-concave in $c_S \in \mathbf{Z}^S$.*
(iii) $F^{\rm NF}_{\mathbf{Z}}$ *is M$^\natural$-concave in $d_P \in \mathbf{Z}^P$ and L$^\natural$-concave in $d_S \in \mathbf{Z}^S$.*

With the aid of the general facts that M$^\natural$-convexity implies supermodularity [16, 17, 18] and L$^\natural$-convexity submodularity, Theorem 1.3 above provides us with a somewhat deeper understanding of (1.2). Namely, it is understood that

$$\begin{array}{llll}
F^{\rm NF} \text{ is L}^\natural\text{-convex}, & \text{hence} & \text{submodular and convex}, & \text{in } w_P, \\
F^{\rm NF} \text{ is M}^\natural\text{-concave}, & \text{hence} & \text{submodular and concave}, & \text{in } c_P \text{ and in } d_P, \\
F^{\rm NF} \text{ is M}^\natural\text{-convex}, & \text{hence} & \text{supermodular and convex}, & \text{in } w_S, \\
F^{\rm NF} \text{ is L}^\natural\text{-concave}, & \text{hence} & \text{supermodular and concave}, & \text{in } c_S \text{ and in } d_S.
\end{array}$$

With economic terms of *substitutes* and *complements* we have the following correspondences:

$$\begin{array}{lcl}
f \text{ is submodular} & \iff & \text{goods are substitutes}, \\
f \text{ is supermodular} & \iff & \text{goods are complements},
\end{array}$$

where f is interpreted as representing a utility function. On the other hand, M$^\natural$-concave functions over the integer lattice provide with a natural model of utility functions in an economy with indivisible commodities (see [14, Section 11.3], [25]). It is shown in [3, 20] that under some appropriate assumptions M$^\natural$-concavity is equivalent to nice properties such as the gross substitutes property [11], and the single improvement condition and the no complementarity condition [9]. From the viewpoint of mathematical economics, Theorems 1.3 and 1.4, together with established results in discrete convex analysis [14], show that the optimal value functions $F^{\rm NF}$ and $F^{\rm NF}_{\mathbf{Z}}$ have nice combinatorial properties such as the gross substitutes property in addition to submodularity and supermodularity.

1.2 Extension to Linear and Separable Concave Programs

Our results can be extended to general linear and nonlinear programs as follows.

We first consider the extension to a more general linear program

$$F^{\rm LP}(w, c, d) = \max\{w^{\rm T}\xi \mid N\xi = \mathbf{0},\ d \le \xi \le c\}, \tag{1.3}$$

where the coefficient matrix N can be any real matrix with rows and columns indexed by V and A, respectively. We also denote by $F^{\rm LP}_{\mathbf{Z}}$ the function $F^{\rm LP}$ restricted to integer parameters $(w, c, d) \in \mathbf{Z}^A \times \mathbf{Z}^A \times \mathbf{Z}^A$. For any distinct elements $a, b \in A$, we say that a and b are *substitutes* (resp., *complements*) if every circuit $\pi \in \mathbf{R}^A$ satisfies $\pi(a) \cdot \pi(b) \le 0$ (resp., $\pi(a) \cdot \pi(b) \ge 0$), where $\pi \in \mathbf{R}^A$ is said to be a *circuit* if it is a nonzero vector such that $N\pi = \mathbf{0}$ and its support $\mathrm{supp}(\pi)$ is minimal. When N is the incidence matrix of a directed graph, elements a and b are substitutes (resp., complements) if and only if the arcs a and b are "parallel" (resp., "series"). For any $A' \subseteq A$, we say that A' is a set of *substitutes* (resp., *complements*) if it consists of pairwise substitutes (resp., complements) elements.

Theorem 1.1 on submodularity and supermodularity is extended as follows.

Theorem 1.5 ([6, 7, 21]). *Let $P \subseteq A$ be a set of substitutes, and $S \subseteq A$ a set of complements.*
(i) $F^{\rm LP}$ *is submodular in w_P, in c_P, and in d_P.*
(ii) $F^{\rm LP}$ *is supermodular in w_S, in c_S, and in d_S.*

Our results can also be extended to the case where the matrix N is totally unimodular. Recall that a matrix N is said to be totally unimodular if any subdeterminant of N is equal to either 0, +1, or −1. The incidence matrix of a directed graph is totally unimodular, and hence the following theorems contain Theorems 1.3 and 1.4 as special cases.

Theorem 1.6. *Let $P \subseteq A$ be a set of substitutes, and $S \subseteq A$ a set of complements. Suppose that the matrix N in* (1.3) *is totally unimodular.*
(i) *F^{LP} is $\mathrm{L}^\natural$-convex in $w_P \in \mathbf{R}^P$ and $\mathrm{M}^\natural$-convex in $w_S \in \mathbf{R}^S$.*
(ii) *F^{LP} is $\mathrm{M}^\natural$-concave in $c_P \in \mathbf{R}^P$ and $\mathrm{L}^\natural$-concave in $c_S \in \mathbf{R}^S$.*
(iii) *F^{LP} is $\mathrm{M}^\natural$-concave in $d_P \in \mathbf{R}^P$ and $\mathrm{L}^\natural$-concave in $d_S \in \mathbf{R}^S$.*

Theorem 1.7. *Let $P \subseteq A$ be a set of substitutes, and $S \subseteq A$ a set of complements. Suppose that the matrix N in* (1.3) *is totally unimodular.*
(i) *$F^{\mathrm{LP}}_{\mathbf{Z}}$ is $\mathrm{L}^\natural$-convex in $w_P \in \mathbf{Z}^P$ and $\mathrm{M}^\natural$-convex in $w_S \in \mathbf{Z}^S$.*
(ii) *$F^{\mathrm{LP}}_{\mathbf{Z}}$ is $\mathrm{M}^\natural$-concave in $c_P \in \mathbf{Z}^P$ and $\mathrm{L}^\natural$-concave in $c_S \in \mathbf{Z}^S$.*
(iii) *$F^{\mathrm{LP}}_{\mathbf{Z}}$ is $\mathrm{M}^\natural$-concave in $d_P \in \mathbf{Z}^P$ and $\mathrm{L}^\natural$-concave in $d_S \in \mathbf{Z}^S$.*

We then consider a further extension to a nonlinear program with a separable concave objective function

$$F^{\mathrm{SC}}(c,d) = \max\{\sum_{a\in A} f_a(\xi_a) \mid N\xi = \mathbf{0},\ d \leq \xi \leq c\}, \tag{1.4}$$

where $f_a : \mathbf{R} \to \mathbf{R}$ ($a \in A$) is a family of univariate concave functions. Since the feasible region is given by a bounded polyhedron, the nonlinear program (1.4) has an optimal solution if the feasible region is nonempty [22, 23, 24]. We also denote by $F^{\mathrm{SC}}_{\mathbf{Z}}$ the function F^{SC} restricted to integer parameters $(c,d) \in \mathbf{Z}^A \times \mathbf{Z}^A$.

Theorem 1.8. *Let $P \subseteq A$ be a set of substitutes, and $S \subseteq A$ a set of complements. Suppose that the matrix N in* (1.4) *is totally unimodular.*
(i) *F^{SC} is $\mathrm{M}^\natural$-concave in $c_P \in \mathbf{R}^P$ and $\mathrm{L}^\natural$-concave in $c_S \in \mathbf{R}^S$.*
(ii) *F^{SC} is $\mathrm{M}^\natural$-concave in $d_P \in \mathbf{R}^P$ and $\mathrm{L}^\natural$-concave in $d_S \in \mathbf{R}^S$.*

Theorem 1.9. *Let $P \subseteq A$ be a set of substitutes, and $S \subseteq A$ a set of complements. Suppose that the matrix N in* (1.4) *is totally unimodular.*
(i) *$F^{\mathrm{SC}}_{\mathbf{Z}}$ is $\mathrm{M}^\natural$-concave in $c_P \in \mathbf{Z}^P$ and $\mathrm{L}^\natural$-concave in $c_S \in \mathbf{Z}^S$.*
(ii) *$F^{\mathrm{SC}}_{\mathbf{Z}}$ is $\mathrm{M}^\natural$-concave in $d_P \in \mathbf{Z}^P$ and $\mathrm{L}^\natural$-concave in $d_S \in \mathbf{Z}^S$.*

Proofs of the theorems above are given in Section 3.

2 Definitions of M-convex and L-convex Functions

2.1 M-convex and L-convex Functions over the Integer Lattice

Let n be a positive integer. A function $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be *M-convex* if $\mathrm{dom}_{\mathbf{Z}}\, f \neq \emptyset$ and f satisfies (M-EXC[$\mathbf{Z}$]):

(M-EXC[Z]) $\forall x, y \in \mathrm{dom}_{\mathbf{Z}} f$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y)$:

$$f(x) + f(y) \geq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j),$$

where

$$\mathrm{dom}_{\mathbf{Z}} f = \{x \in \mathbf{Z}^n \mid f(x) < +\infty\},$$
$$\mathrm{supp}^+(x) = \{i \mid x(i) > 0\}, \quad \mathrm{supp}^-(x) = \{i \mid x(i) < 0\} \qquad (x \in \mathbf{R}^n),$$
$$\chi_i \in \{0,1\}^n\text{: the } i\text{-th unit vector } (i = 1, 2, \ldots, n).$$

A function $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be *M$^\natural$-convex* if the function $\widehat{f} : \mathbf{Z}^n \times \mathbf{Z} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\widehat{f}(x, x_0) = \begin{cases} f(x) & ((x, x_0) \in \mathbf{Z}^n \times \mathbf{Z},\ x_0 = -\sum_{i=1}^n x(i)), \\ +\infty & (\text{otherwise}) \end{cases}$$

is M-convex. M$^\natural$-convexity of a function f is characterized by the following property [15, Theorem 4.2]:

(M$^\natural$-EXC[Z]) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y) \cup \{0\}$:

$$f(x) + f(y) \geq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j),$$

where $\chi_0 = \mathbf{0}$ by convention. A function $f : \mathbf{Z}^n \to \mathbf{R} \cup \{-\infty\}$ is said to be *M-concave* (resp., *M$^\natural$-concave*) if $-f$ is M-convex (resp., M$^\natural$-convex).

On the other hand, a function $g : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be *L-convex* if $\mathrm{dom}_{\mathbf{Z}}\, g \neq \emptyset$ and g satisfies (LF1[**Z**]) and (LF2[**Z**]):

(LF1[Z]) g is submodular, i.e., $g(p) + g(q) \geq g(p \vee q) + g(p \wedge q) \qquad (\forall p, q \in \mathbf{Z}^n)$,
(LF2[Z]) $\exists r \in \mathbf{R}$ such that $g(p + \lambda \mathbf{1}) = g(p) + \lambda r \quad (\forall p \in \mathbf{Z}^n,\ \lambda \in \mathbf{Z})$,

where $p \vee q, p \wedge q \in \mathbf{R}^n$ are vectors defined by

$$(p \vee q)(i) = \max\{p(i), q(i)\}, \qquad (p \wedge q)(i) = \min\{p(i), q(i)\} \qquad (i = 1, 2, \ldots, n).$$

A function $g : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ is called *L$^\natural$-convex* if the function $\widehat{g} : \mathbf{Z}^n \times \mathbf{Z} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\widehat{g}(p, p_0) = g(p - p_0 \mathbf{1}) \qquad ((p, p_0) \in \mathbf{Z}^n \times \mathbf{Z})$$

is L-convex. A function $g : \mathbf{Z}^n \to \mathbf{R} \cup \{-\infty\}$ is said to be *L-concave* (resp., *L$^\natural$-concave*) if $-g$ is L-convex (resp., L$^\natural$-convex).

L$^\natural$-convexity implies submodularity by definition, whereas M$^\natural$-convexity implies supermodularity.

Proposition 2.1 ([17, Theorem 3.8]). *An M$^\natural$-convex function $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ satisfies the supermodular inequality:*

$$f(x) + f(y) \leq f(x \vee y) + f(x \wedge y) \qquad (\forall x, y \in \mathbf{Z}^n).$$

2.2 Closed Proper M-convex and L-convex Functions

A function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be *closed proper convex* if it is a convex function such that the effective domain $\mathrm{dom}\, f = \{x \in \mathbf{R}^n \mid f(x) < +\infty\}$ is nonempty and the epigraph $\{(x, \alpha) \in \mathbf{R}^n \times \mathbf{R} \mid \alpha \geq f(x)\}$ is a closed set [22, 24]. A function $f : \mathbf{R}^n \to \mathbf{R} \cup \{-\infty\}$ is said to be *closed proper concave* if $-f$ is closed proper convex.

A closed proper convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be *M-convex* if $\mathrm{dom}\, f \neq \emptyset$ and f satisfies (M-EXC[**R**]):

(M-EXC[R]) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x - y)$, $\exists j \in \mathrm{supp}^-(x - y)$, $\exists \alpha_0 > 0$:

$$f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \qquad (\forall \alpha \in [0, \alpha_0]),$$

where $[0, \alpha_0] = \{\alpha \in \mathbf{R} \mid 0 \leq \alpha \leq \alpha_0\}$. A closed proper convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be *M$^\natural$-convex* if the function $\widehat{f} : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\widehat{f}(x, x_0) = \begin{cases} f(x) & ((x, x_0) \in \mathbf{R}^n \times \mathbf{R},\ x_0 = -\sum_{i=1}^n x(i)), \\ +\infty & (\text{otherwise}) \end{cases}$$

is M-convex. M$^\natural$-convexity of a closed proper convex function f is characterized by the following property [19, Theorem 2.3]:

(M$^\natural$-EXC[R]) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x - y)$, $\exists j \in \mathrm{supp}^-(x - y) \cup \{0\}$, $\exists \alpha_0 > 0$:

$$f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \qquad (\forall \alpha \in [0, \alpha_0]).$$

A closed proper concave function $f : \mathbf{R}^n \to \mathbf{R} \cup \{-\infty\}$ is said to be *M-concave* (resp., *M$^\natural$-concave*) if $-f$ is M-convex (resp., M$^\natural$-convex).

On the other hand, a closed proper convex function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be *L-convex* if $\mathrm{dom}\, g \neq \emptyset$ and g satisfies (LF1) and (LF2):

(LF1[R]) g is submodular, i.e., $g(p) + g(q) \geq g(p \vee q) + g(p \wedge q) \qquad (\forall p, q \in \mathbf{R}^n)$,
(LF2[R]) $\exists r \in \mathbf{R}$ such that $g(p + \lambda \mathbf{1}) = g(p) + \lambda r \quad (\forall p \in \mathbf{R}^n,\ \lambda \in \mathbf{R})$;

g is called *L$^\natural$-convex* if the function $\widehat{g} : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\widehat{g}(p, p_0) = g(p - p_0 \mathbf{1}) \qquad ((p, p_0) \in \mathbf{R}^n \times \mathbf{R})$$

is L-convex. A closed proper concave function $g : \mathbf{R}^n \to \mathbf{R} \cup \{-\infty\}$ is said to be *L-concave* (resp., *L$^\natural$-concave*) if $-g$ is L-convex (resp., L$^\natural$-convex).

Closed proper L$^\natural$-convexity implies submodularity by definition, whereas closed proper M$^\natural$-convexity implies supermodularity.

Proposition 2.2 ([18, Proposition 3.4]). *A closed proper M$^\natural$-convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ satisfies the supermodular inequality:*

$$f(x) + f(y) \leq f(x \vee y) + f(x \wedge y) \qquad (\forall x, y \in \mathbf{R}^n).$$

3 Proofs

In this section we prove Theorems 1.6 and 1.8. Theorems 1.7 and 1.9 can be proven in the same way, and Theorems 1.3 and 1.4 are immediate corollaries of Theorems 1.6 and 1.7, respectively. The proof of Theorem 1.8 relies on the property of univariate concave functions that

$$f_a(\eta) + f_a(\zeta) \leq f_a(\eta+\delta) + f_a(\zeta-\delta) \quad (\forall \eta, \zeta \in \mathbf{R} \text{ with } \eta < \zeta,\ 0 \leq \forall \delta \leq \zeta - \eta). \tag{3.1}$$

In the following, we show the properties of F^{LP} and F^{SC} w.r.t. the weight w and the upper bound c. Then, the properties of F^{LP} and F^{SC} w.r.t. the lower bound d can be shown as follows. By the definition, a function $f(x)$ is M$^\natural$-concave (resp., L$^\natural$-concave) in x if and only if $f(-x')$ is M$^\natural$-concave (resp., L$^\natural$-concave) in x'. Since the optimal value F^{SC} is rewritten as

$$F^{\mathrm{SC}} = \max\{\sum_{a \in A} f_a(\xi_a) \mid N\xi = \mathbf{0},\ d \leq \xi \leq c\} = \max\{\sum_{a \in A} f_a(-\xi'_a) \mid N\xi' = \mathbf{0},\ -c \leq \xi' \leq -d\},$$

M$^\natural$-concavity in d_P (resp., L$^\natural$-concavity in d_S) follows immediately from M$^\natural$-concavity in c_P (resp., L$^\natural$-concavity in c_S).

To the end of this section we assume that N is a totally unimodular matrix. Then, any circuit is a multiple of a $\{0, +1, -1\}$ vector, and accordingly, we assume in the following that every circuit is a $\{0, +1, -1\}$ vector.

3.1 Basic Properties of Sets of Substitutes and Complements

We start with basic properties of sets of substitutes and complements that we use in the proof. The main technical tool in the proof is the *conformal decomposition* (see, e.g., [1, 10, 23]) of a vector ξ with $N\xi = \mathbf{0}$, which is a representation of ξ as a positive sum of circuits conformal to ξ, i.e.,

$$\xi = \sum_{i=1}^{m} \beta_i \pi_i,$$

where $\beta_i > 0$ and $\pi_i : A \to \mathbf{R}$ is a circuit with $\mathrm{supp}^+(\pi_i) \subseteq \mathrm{supp}^+(\xi)$ and $\mathrm{supp}^-(\pi_i) \subseteq \mathrm{supp}^-(\xi)$ for $i = 1, 2, \ldots, m$.

Proposition 3.1. *Let π be a circuit.*
(i) $|\mathrm{supp}^+(\pi) \cap P| \leq 1$ *and* $|\mathrm{supp}^-(\pi) \cap P| \leq 1$ *for any set P of substitutes.*
(ii) $|\mathrm{supp}^+(\pi) \cap S| = 0$ *or* $|\mathrm{supp}^-(\pi) \cap S| = 0$ *for any set S of complements.*

Proposition 3.2. *Let S be a set of complements, and ξ and $\widetilde{\xi}$ be optimal solutions of the linear program of the form* (1.3)*. Then, there exist some optimal solutions $\xi_\wedge$ and $\xi_\vee$ such that*

$$\xi_\wedge(a) = \min\{\xi(a), \widetilde{\xi}(a)\}, \qquad \xi_\vee(a) = \max\{\xi(a), \widetilde{\xi}(a)\} \qquad (a \in S). \tag{3.2}$$

Proof. Consider a conformal decomposition $\sum_{i=1}^{m} \beta_i \pi_i$ of a vector $\widetilde{\xi} - \xi$, where we assume that $\mathrm{supp}^+(\pi_i) \cap S \neq \emptyset$ for $i = 1, 2, \ldots, \ell$ and $\mathrm{supp}^+(\pi_i) \cap S = \emptyset$ for $i = \ell+1, \ell+2, \ldots, m$. Then, it follows from Proposition 3.1 (ii) that $\mathrm{supp}^-(\pi_i) \cap S = \emptyset$ for $i = 1, 2, \ldots, \ell$. Therefore, $\xi_\wedge = \xi + \sum_{i=\ell+1}^{m} \beta_i \pi_i$ and $\xi_\vee = \xi + \sum_{i=1}^{\ell} \beta_i \pi_i$ are feasible solutions satisfying the condition (3.2). Since $\xi + \beta_i \pi_i$ and $\widetilde{\xi} - \beta_i \pi_i$ are feasible solutions, we have $w^{\mathrm{T}} \pi_i = 0$ for all $i = 1, 2, \ldots, m$. Hence, $\xi_\wedge$ and $\xi_\vee$ are optimal solutions. □

Proposition 3.3. *Let S be a set of complements, and π_1 and π_2 be circuits. If $\mathrm{supp}^+(\pi_1)\cap\mathrm{supp}^+(\pi_2)\cap S\neq\emptyset$, then there exists a circuit π such that*

$$\mathrm{supp}^+(\pi)\subseteq\mathrm{supp}^+(\pi_1)\cup\mathrm{supp}^+(\pi_2),\qquad \mathrm{supp}^-(\pi)\subseteq\mathrm{supp}^-(\pi_1)\cup\mathrm{supp}^-(\pi_2),$$
$$\mathrm{supp}^+(\pi)\cap S=(\mathrm{supp}^+(\pi_1)\cup\mathrm{supp}^+(\pi_2))\cap S.$$

Proof. We denote $S_i^+=\mathrm{supp}^+(\pi_i)$, $S_i^-=\mathrm{supp}^-(\pi_i)$, and $S_i=\mathrm{supp}(\pi_i)$ for $i=1,2$. By Proposition 3.1 (ii) and $S_1^+\cap S_2^+\cap S\neq\emptyset$, we have $S_1\cap S=S_1^+\cap S$ and $S_2\cap S=S_2^+\cap S$. Let $a\in(S_2^+\setminus S_1^+)\cap S$. We have $a\notin S_1^-$, and therefore $a\notin S_1$. In the following, we show that there exists a circuit π' such that

$$\mathrm{supp}^+(\pi')\subseteq S_1^+\cup S_2^+,\quad \mathrm{supp}^-(\pi')\subseteq S_1^-\cup S_2^-,\tag{3.3}$$
$$\mathrm{supp}^+(\pi')\cap S\supseteq(S_1^+\cup\{a\})\cap S.\tag{3.4}$$

Repeating this we can find π.

By the conformal decomposition of $\pi_2-\pi_1$, there exists some circuit $\hat{\pi}$ such that

$$a\in\mathrm{supp}^+(\hat{\pi})\subseteq S_2^+\cup S_1^-,\qquad \mathrm{supp}^-(\hat{\pi})\subseteq S_2^-\cup S_1^+.\tag{3.5}$$

We assume that $\mathrm{supp}(\hat{\pi})\setminus S_1$ is minimal among all such circuits. Put $\pi'=\pi_1+\hat{\pi}$. Then, π' is a circuit, as shown later. From (3.5) we have the condition (3.3) and $a\in\mathrm{supp}^+(\pi')\cap S$. Since $a\in\mathrm{supp}^+(\hat{\pi})\cap S$, we have $\mathrm{supp}^-(\hat{\pi})\cap S=\emptyset$ by Proposition 3.1 (ii). Therefore, $S_1^+\cap S=(S_1^+\setminus\mathrm{supp}^-(\hat{\pi}))\cap S\subseteq\mathrm{supp}^+(\pi')\cap S$, implying the condition (3.4).

We now prove that π' is a circuit. Since $N\pi'=\mathbf{0}$, there exists some circuit π'' satisfying

$$a\in\mathrm{supp}^+(\pi'')\subseteq\mathrm{supp}^+(\pi'),\quad \mathrm{supp}^-(\pi'')\subseteq\mathrm{supp}^-(\pi').\tag{3.6}$$

We will show that $\pi''=\pi'$.

Claim 1: $\mathrm{supp}(\pi'-\pi'')\subseteq S_1$.
[Proof of Claim] The vector $\pi''-\pi_1$ satisfies $N(\pi''-\pi_1)=\mathbf{0}$, $a\in\mathrm{supp}^+(\pi''-\pi_1)\subseteq S_2^+\cup S_1^-$, and $\mathrm{supp}^-(\pi''-\pi_1)\subseteq S_2^-\cup S_1^+$. For $b\in A\setminus S_1$ we have $\hat{\pi}(b)=\pi'(b)-\pi_1(b)=\pi'(b)$ and $\pi''(b)-\pi_1(b)=\pi''(b)$. Therefore, it follows from (3.6) that

$$\mathrm{supp}^+(\pi''-\pi_1)\setminus S_1\subseteq\mathrm{supp}^+(\hat{\pi})\setminus S_1,\quad \mathrm{supp}^-(\pi''-\pi_1)\setminus S_1\subseteq\mathrm{supp}^-(\hat{\pi})\setminus S_1.$$

Hence, the choice of $\hat{\pi}$ implies that $\mathrm{supp}(\pi''-\pi_1)\setminus S_1=\mathrm{supp}(\hat{\pi})\setminus S_1$, from which follows $\pi''(b)=\pi''(b)-\pi_1(b)=\hat{\pi}(b)=\pi'(b)$ for all $b\in A\setminus S_1$. [End of Claim]

Claim 2: $\mathrm{supp}(\pi')\cap S_1\subsetneq S_1$.
[Proof of Claim] We have $\mathrm{supp}(\hat{\pi})\subseteq S_1\cup S_2$ by (3.5). Since $S_1\cap S_2\neq\emptyset$ by the assumption and π_2 is a circuit, we have $\mathrm{supp}(\hat{\pi})\cap S_1\neq\emptyset$. This implies $\mathrm{supp}(\pi')\cap S_1\subsetneq S_1$ since $\pi'=\pi_1+\hat{\pi}$.
[End of Claim]

It follows from $\mathrm{supp}(\pi'')\subseteq\mathrm{supp}(\pi')$ and Claims 1 and 2 that $\mathrm{supp}(\pi'-\pi'')\subseteq\mathrm{supp}(\pi')\cap S_1\subsetneq S_1$. This implies $\pi'-\pi''=\mathbf{0}$ since $N(\pi'-\pi'')=\mathbf{0}$ and π_1 is a circuit. This completes the proof. ☐

Proposition 3.4. *Let S be a set of complements. For any vector ξ with $N\xi = \mathbf{0}$ and $a_* \in S \backslash \mathrm{supp}^-(\xi)$, there exists a conformal decomposition $\sum_{i=1}^{m} \beta_i \pi_i$ of ξ and an integer ℓ with $0 \leq \ell \leq m$ such that*

$$a_* \in \mathrm{supp}^+(\pi_1) \cap S \subseteq \mathrm{supp}^+(\pi_2) \cap S \subseteq \cdots \subseteq \mathrm{supp}^+(\pi_\ell) \cap S, \quad \pi_i(a_*) = 0 \ (i = \ell+1, \ell+2, \ldots, m). \tag{3.7}$$

Proof. If $\xi(a_*) = 0$, then any conformal decomposition of ξ satisfies the condition (3.7) with $\ell = 0$. Otherwise, put

$$\Pi = \{\pi \mid \pi : \text{circuit},\ a_* \in \mathrm{supp}^+(\pi) \subseteq \mathrm{supp}^+(\xi),\ \mathrm{supp}^-(\pi) \subseteq \mathrm{supp}^-(\xi)\}.$$

Let π_* be a circuit in Π such that $\mathrm{supp}^+(\pi_*) \cap S$ is maximal among all circuits in Π. By Proposition 3.3, we have $\mathrm{supp}^+(\pi) \cap S \subseteq \mathrm{supp}^+(\pi_*) \cap S$ for all $\pi \in \Pi$. Let $\beta = \min\{|\xi(a)| \mid a \in \mathrm{supp}(\pi_*)\}$ (> 0), and put $\xi' = \xi - \beta\pi_*$. Then, we have $\mathrm{supp}^+(\xi') \subseteq \mathrm{supp}^+(\xi)$, $\mathrm{supp}^-(\xi') \subseteq \mathrm{supp}^-(\xi)$, and there exists some $b \in \mathrm{supp}(\xi)$ with $\xi'(b) = 0$. Repeating this argument, we can find a conformal decomposition satisfying (3.7). $\square$

3.2 Proof of L$^\natural$-convexity in w_P

We prove the L$^\natural$-convexity of the function F^{LP} in w_P, the former part of Theorem 1.6 (i).

We denote $F = F^{\mathrm{LP}}$ for simplicity. L$^\natural$-convexity of F in w_P is equivalent to submodularity of $F(w - w_0\chi_P, c, d)$ in (w_P, w_0), which in turn is equivalent to

$$F(w + \lambda\chi_a, c, d) + F(w + \mu\chi_b, c, d) \geq F(w, c, d) + F(w + \lambda\chi_a + \mu\chi_b, c, d), \tag{3.8}$$

$$F(w + \lambda\chi_a, c, d) + F(w - \mu\chi_P, c, d) \geq F(w, c, d) + F(w + \lambda\chi_a - \mu\chi_P, c, d) \tag{3.9}$$

for $a, b \in P$ with $a \neq b$ and $\lambda, \mu \in \mathbf{R}_+$, where $\chi_P \in \{0,1\}^A$ denotes the characteristic vector of $P \subseteq A$.

To show (3.8) let ξ and $\widetilde{\xi}$ be optimal solutions for w and $w + \lambda\chi_a + \mu\chi_b$. We can establish (3.8) by constructing feasible solutions ξ_a and ξ_b such that

$$\xi_a + \xi_b = \xi + \widetilde{\xi}, \qquad \lambda[\xi_a(a) - \widetilde{\xi}(a)] + \mu[\xi_b(b) - \widetilde{\xi}(b)] \geq 0, \tag{3.10}$$

since this implies

$$(w + \lambda\chi_a)^{\mathrm{T}}\xi_a + (w + \mu\chi_b)^{\mathrm{T}}\xi_b \geq w^{\mathrm{T}}\xi + (w + \lambda\chi_a + \mu\chi_b)^{\mathrm{T}}\widetilde{\xi},$$

of which the left-hand side is bounded by $F(w+\lambda\chi_a, c, d) + F(w+\mu\chi_b, c, d)$ and the right-hand side is equal to $F(w, c, d) + F(w+\lambda\chi_a+\mu\chi_b, c, d)$. If $\widetilde{\xi}(a) \leq \xi(a)$, we can take $\xi_a = \xi$ and $\xi_b = \widetilde{\xi}$ to meet (3.10). If $\widetilde{\xi}(b) \leq \xi(b)$, we can take $\xi_a = \widetilde{\xi}$ and $\xi_b = \xi$ to meet (3.10). Otherwise, we make use of the conformal decomposition $\widetilde{\xi} - \xi = \sum_{i=1}^{m} \beta_i \pi_i$. Since $a \in \mathrm{supp}^+(\widetilde{\xi} - \xi)$ we may assume $\pi_i(a) > 0$ for $i = 1, 2, \ldots, \ell$ and $\pi_i(a) = 0$ for $i = \ell+1, \ell+2, \ldots, m$. We have $\pi_i(b) = 0$ for $i = 1, 2, \ldots, \ell$ by Proposition 3.1 (i), since P is a set of substitutes and $\{a, b\} \subseteq \mathrm{supp}^+(\widetilde{\xi} - \xi)$. Therefore, $\xi_a = \xi + \sum_{i=1}^{\ell} \beta_i \pi_i$ and $\xi_b = \xi + \sum_{i=\ell+1}^{m} \beta_i \pi_i$ are feasible solutions that satisfy (3.10).

To show (3.9) let ξ and $\widetilde{\xi}$ be optimal solutions for w and $w + \lambda\chi_a - \mu\chi_P$. We can establish (3.9) by constructing feasible solutions ξ_a and ξ_P such that

$$\xi_a + \xi_P = \xi + \widetilde{\xi}, \qquad \lambda[\xi_a(a) - \widetilde{\xi}(a)] + \mu[\sum_{e \in P} \widetilde{\xi}(e) - \sum_{e \in P} \xi_P(e)] \geq 0 \tag{3.11}$$

since this implies

$$(w + \lambda\chi_a)^{\mathrm{T}}\xi_a + (w - \mu\chi_P)^{\mathrm{T}}\xi_P \geq w^{\mathrm{T}}\xi + (w + \lambda\chi_a - \mu\chi_P)^{\mathrm{T}}\widetilde{\xi}.$$

If $\widetilde{\xi}(a) \leq \xi(a)$, we can take $\xi_a = \xi$ and $\xi_P = \widetilde{\xi}$ to meet (3.11). Otherwise we use the conformal decomposition $\widetilde{\xi} - \xi = \sum_{i=1}^{m} \beta_i\pi_i$, in which we assume $\pi_i(a) > 0$ for $i = 1, 2, \ldots, \ell$ and $\pi_i(a) = 0$ for $i = \ell+1, \ell+2, \ldots, m$. Since P is a set of substitutes we have $|\mathrm{supp}^-(\pi_i) \cap P| \leq 1$ by Proposition 3.1 (i), and hence $\sum_{e\in P} \pi_i(e) \geq 0$ for $i = 1, 2, \ldots, \ell$. Therefore, $\xi_a = \xi + \sum_{i=1}^{\ell} \beta_i\pi_i$ and $\xi_P = \xi + \sum_{i=\ell+1}^{m} \beta_i\pi_i$ are feasible solutions that satisfy (3.11).

3.3 Proof of M$^\natural$-concavity in c_P

We prove the M$^\natural$-concavity of the function F^{SC} in c_P, the former part of Theorem 1.8 (i). This contains the former part of Theorem 1.6 (ii) as a special case.

We denote $F = F^{\mathrm{SC}}$ for simplicity. We prove the M$^\natural$-concavity of F in c_P by establishing (M$^\natural$-EXC[$\mathbf{R}$]) for $-F$ as a function in c_P. In our notation this reads as follows:

> Let $c_1, c_2 \in \mathbf{R}^A$ be capacities with $c_1(e) = c_2(e)$ for all $e \in A \setminus P$. For each $a \in \mathrm{supp}^+(c_1 - c_2)$, there exist $b \in \mathrm{supp}^-(c_1 - c_2) \cup \{0\}$ and a positive number α_0 such that
>
> $$F(c_1, d) + F(c_2, d) \leq F(c_1 - \alpha(\chi_a - \chi_b), d) + F(c_2 + \alpha(\chi_a - \chi_b), d) \qquad (\forall\alpha \in [0, \alpha_0]).$$

Let ξ_1 and ξ_2 be optimal solutions for c_1 and c_2, respectively. We shall find $\alpha_0 > 0$ and $b \in \mathrm{supp}^-(c_1 - c_2) \cup \{0\}$ such that, for any $\alpha \in [0, \alpha_0]$, there exist vectors ξ_1' and ξ_2' with $N\xi_1' = N\xi_2' = \mathbf{0}$ such that

$$\left.\begin{array}{l} \xi_1' + \xi_2' = \xi_1 + \xi_2, \\ d \leq \xi_1' \leq c_1 - \alpha(\chi_a - \chi_b), \quad d \leq \xi_2' \leq c_2 + \alpha(\chi_a - \chi_b), \\ \xi_1 \wedge \xi_2 \leq \xi_1' \wedge \xi_2' \leq \xi_1' \vee \xi_2' \leq \xi_1 \vee \xi_2. \end{array}\right\} \tag{3.12}$$

If $\xi_1(a) < c_1(a)$, we can take $\alpha_0 = c_1(a) - \xi_1(a)$, $b = 0$, $\xi_1' = \xi_1$ and $\xi_2' = \xi_2$ to meet (3.12). Suppose $\xi_1(a) = c_1(a)$. We have $\xi_1(a) = c_1(a) > c_2(a) \geq \xi_2(a)$. Let π be a circuit such that $a \in \mathrm{supp}^+(\pi) \subseteq \mathrm{supp}^+(\xi_1 - \xi_2)$ and $\mathrm{supp}^-(\pi) \subseteq \mathrm{supp}^-(\xi_1 - \xi_2)$. Since P is a set of substitutes and $a \in \mathrm{supp}^+(\pi)$, we have $\mathrm{supp}^+(\pi) \cap P = \{a\}$ and $|\mathrm{supp}^-(\pi) \cap P| \leq 1$ by Proposition 3.1 (i). If $|\mathrm{supp}^-(\pi) \cap P| = 1$, define b by $\{b\} = \mathrm{supp}^-(\pi) \cap P$; otherwise put $b = 0$. We put $\alpha_0 = \min\{|\xi_1(e) - \xi_2(e)| \mid e \in \mathrm{supp}(\pi)\}$. Then $\xi_1' = \xi_1 - \alpha\pi$ and $\xi_2' = \xi_2 + \alpha\pi$ satisfy (3.12) if $\alpha \in [0, \alpha_0]$.

3.4 Proof of M$^\natural$-convexity in w_S

We prove the M$^\natural$-convexity of the function F^{LP} in w_S, the latter part of Theorem 1.6 (i).

We denote $F = F^{\mathrm{LP}}$ for simplicity. We prove the M$^\natural$-convexity of F in w_S by establishing (M$^\natural$-EXC[$\mathbf{R}$]). In our notation this reads as follows:

> Let $w_1, w_2 \in \mathbf{R}^A$ be weights with $w_1(e) = w_2(e)$ for all $e \in A \setminus S$. For each $a \in \mathrm{supp}^+(w_1 - w_2)$, there exist $b \in \mathrm{supp}^-(w_1 - w_2) \cup \{0\}$ and a positive number α_0 such that
>
> $$F(w_1, c, d) + F(w_2, c, d) \geq F(w_1 - \alpha(\chi_a - \chi_b), c, d) + F(w_2 + \alpha(\chi_a - \chi_b), c, d) \qquad (\forall\alpha \in [0, \alpha_0]).$$

Let ξ_1 and ξ_2 be optimal solutions for w_1 and w_2, respectively, with $\xi_1(a)$ minimum and $\xi_2(a)$ maximum.

Proposition 3.5. *There exists $\alpha_0 > 0$ such that ξ_1 is optimal for $w_1 - \alpha\chi_a$ and ξ_2 is optimal for $w_2 + \alpha\chi_a$ for all $\alpha \in [0, \alpha_0]$.*

Proof. For any circuit π such that $\pi(a) = -1$ and $d \le \xi_1 + \beta\pi \le c$ for some $\beta > 0$, we have $w_1^{\mathrm{T}}(\xi_1+\beta\pi) < w_1^{\mathrm{T}}\xi_1$ by the choice of ξ_1. Hence, we have $w_1^{\mathrm{T}}\pi < 0$ for any such circuit π. Let $\alpha_1 > 0$ be the minimum of $-w_1^{\mathrm{T}}\pi$ over all such circuits π; if there exists no such circuit π, then we put $\alpha_1 = +\infty$. Then, ξ_1 is optimal for $w_1 - \alpha\chi_a$ for all $\alpha \in [0, \alpha_1]$, since $(w_1 - \alpha\chi_a)^{\mathrm{T}}(\xi_1 + \beta\pi) \le (w_1 - \alpha\chi_a)^{\mathrm{T}}\xi_1$ for any $\beta > 0$ and circuit π such that $d \le \xi_1 + \beta\pi \le c$. Similarly, let $\alpha_2 > 0$ be the minimum of $-w_2^{\mathrm{T}}\pi$ over all circuits π such that $\pi(a) = 1$ and $d \le \xi_2 + \beta\pi \le c$ for some $\beta > 0$. Then ξ_2 is optimal for $w_2 + \alpha\chi_a$ for all $\alpha \in [0, \alpha_2]$. Put $\alpha_0 = \min(\alpha_1, \alpha_2)$. □

If $\xi_1(a) \ge \xi_2(a)$, we can take $b = 0$ in (M$^\natural$-EXC[$\mathbf{R}$]), since

$$\begin{aligned} F(w_1, c, d) + F(w_2, c, d) &= w_1^{\mathrm{T}}\xi_1 + w_2^{\mathrm{T}}\xi_2 \\ &\ge (w_1 - \alpha\chi_a)^{\mathrm{T}}\xi_1 + (w_2 + \alpha\chi_a)^{\mathrm{T}}\xi_2 = F(w_1 - \alpha\chi_a, c, d) + F(w_2 + \alpha\chi_a, c, d), \end{aligned}$$

where the last equality is by Proposition 3.5. In what follows we assume $\xi_1(a) < \xi_2(a)$.

By Proposition 3.2, we can impose further conditions on ξ_1 and ξ_2 that, for each $b \in S \setminus \{a\}$, $\xi_1(b)$ is maximum among all optimal ξ_1 for w_1 with $\xi_1(a)$ minimum, and $\xi_2(b)$ is minimum among all optimal ξ_2 for w_2 with $\xi_2(a)$ maximum.

Proposition 3.6. *There exists $\alpha_0 > 0$ such that ξ_1 is optimal for $w_1 - \alpha(\chi_a - \chi_b)$ and ξ_2 is optimal for $w_2 + \alpha(\chi_a - \chi_b)$ for all $b \in S \setminus \{a\}$ and for all $\alpha \in [0, \alpha_0]$.*

Proof. For any circuit π such that $\pi(a) - \pi(b) = -1$ for some $b \in S \setminus \{a\}$ and $d \le \xi_1 + \beta\pi \le c$ for some $\beta > 0$, we have $w_1^{\mathrm{T}}(\xi_1 + \beta\pi) < w_1^{\mathrm{T}}\xi_1$ by the choice of ξ_1. Hence, we have $w_1^{\mathrm{T}}\pi < 0$ for any such circuit π. Let $\alpha_1 > 0$ be the minimum of $-w_1^{\mathrm{T}}\pi$ over all such circuits π. Then ξ_1 is optimal for $w_1 - \alpha(\chi_a - \chi_b)$ for all $\alpha \in [0, \alpha_1]$. Similarly, let $\alpha_2 > 0$ be the minimum of $-w_2^{\mathrm{T}}\pi$ over all circuits π such that $\pi(a) - \pi(b) = 1$ for some $b \in S \setminus \{a\}$ and $d \le \xi_2 + \beta\pi \le c$ for some $\beta > 0$. Then ξ_2 is optimal for $w_2 + \alpha(\chi_a - \chi_b)$ for all $\alpha \in [0, \alpha_2]$. Put $\alpha_0 = \min(\alpha_1, \alpha_2)$. □

Proposition 3.6 implies that for all $b \in S \setminus \{a\}$ and $\alpha \in [0, \alpha_0]$ we have

$$\begin{aligned} F(w_1, c, d) + F(w_2, c, d) &- F(w_1 - \alpha(\chi_a - \chi_b), c, d) - F(w_2 + \alpha(\chi_a - \chi_b), c, d) \\ &= w_1^{\mathrm{T}}\xi_1 + w_2^{\mathrm{T}}\xi_2 - (w_1 - \alpha(\chi_a - \chi_b))^{\mathrm{T}}\xi_1 - (w_2 + \alpha(\chi_a - \chi_b))^{\mathrm{T}}\xi_2 \\ &= \alpha[(\xi_2(b) - \xi_1(b)) - (\xi_2(a) - \xi_1(a))]. \end{aligned} \tag{3.13}$$

We want to find $b \in \mathrm{supp}^-(w_1 - w_2)$ for which (3.13) is nonnegative.

We make use of the conformal decomposition $\xi_2 - \xi_1 = \sum_{i=1}^m \beta_i\pi_i$. Since S is a set of complements we may assume, by Proposition 3.4, that

$$a \in \mathrm{supp}^+(\pi_1) \cap S \subseteq \mathrm{supp}^+(\pi_2) \cap S \subseteq \cdots \subseteq \mathrm{supp}^+(\pi_\ell) \cap S$$

and $\pi_i(a) = 0$ for $i = \ell+1, \ell+2, \ldots, m$; then $\mathrm{supp}^-(\pi_i) \cap S = \emptyset$ for $i = 1, 2, \ldots, \ell$.

Proposition 3.7. *There exists* $b \in (\mathrm{supp}^+(\pi_1) \cap S) \cap \mathrm{supp}^-(w_1 - w_2)$.

Proof. We have $w_1^{\mathrm{T}}\pi_1 \leq 0$, since ξ_1 is optimal for w_1 and $d \leq \xi_1 + \beta_1\pi_1 \leq c$. Similarly, we have $-w_2^{\mathrm{T}}\pi_1 \leq 0$. Hence

$$0 \geq (w_1 - w_2)^{\mathrm{T}}\pi_1 = \sum_{b\in S}(w_1(b) - w_2(b))\pi_1(b) = \sum_{b\in \mathrm{supp}^+(\pi_1)\cap S}(w_1(b) - w_2(b)).$$

Since $w_1(a) - w_2(a) > 0$ in this summation, we must have $w_1(b) - w_2(b) < 0$ for some $b \in \mathrm{supp}^+(\pi_1) \cap S$. $\square$

For $b \in (\mathrm{supp}^+(\pi_1) \cap S) \cap \mathrm{supp}^-(w_1 - w_2)$ in Proposition 3.7, we have

$$\xi_2(b) - \xi_1(b) = \sum_{i=1}^{\ell}\beta_i + \sum_{i=\ell+1}^{m}\beta_i\pi_i(b) \geq \sum_{i=1}^{\ell}\beta_i = \xi_2(a) - \xi_1(a),$$

which shows the nonnegativity of (3.13).

3.5 Proof of L$^\natural$-concavity in c_S

We prove the L$^\natural$-concavity of the function F^{SC} in c_S, the latter part of Theorem 1.8 (i). This contains the latter part of Theorem 1.6 (ii) as a special case.

We denote $F = F^{\mathrm{SC}}$ for simplicity. L$^\natural$-concavity of F in c_S is equivalent to supermodularity of $F(c - c_0\chi_S, d)$ in (c_S, c_0), which in turn is equivalent to

$$F(c + \lambda\chi_a, d) + F(c + \mu\chi_b, d) \leq F(c, d) + F(c + \lambda\chi_a + \mu\chi_b, d), \tag{3.14}$$

$$F(c + \lambda\chi_a, d) + F(c - \mu\chi_S, d) \leq F(c, d) + F(c + \lambda\chi_a - \mu\chi_S, d) \tag{3.15}$$

for $a, b \in S$ with $a \neq b$ and $\lambda, \mu \in \mathbf{R}_+$, where $\chi_S \in \{0, 1\}^A$ denotes the characteristic vector of $S \subseteq A$.

To show (3.14) let ξ_a and ξ_b be optimal solutions for $c + \lambda\chi_a$ and $c + \mu\chi_b$. We can establish (3.14) by constructing vectors ξ and $\widetilde{\xi}$ with $N\xi = N\widetilde{\xi} = \mathbf{0}$ such that

$$\left.\begin{array}{l} \xi + \widetilde{\xi} = \xi_a + \xi_b, \\ d \leq \xi \leq c, \quad d \leq \widetilde{\xi} \leq c + \lambda\chi_a + \mu\chi_b, \\ \xi_a \wedge \xi_b \leq \xi \wedge \widetilde{\xi} \leq \xi \vee \widetilde{\xi} \leq \xi_a \vee \xi_b. \end{array}\right\} \tag{3.16}$$

If $\xi_a(a) \leq c(a)$, we can take $\xi = \xi_a$ and $\widetilde{\xi} = \xi_b$ to meet (3.16). If $\xi_b(b) \leq c(b)$, we can take $\xi = \xi_b$ and $\widetilde{\xi} = \xi_a$ to meet (3.16). Otherwise, we have $\xi_a(a) > c(a) \geq \xi_b(a)$ and $\xi_a(b) \leq c(b) < \xi_b(b)$, and therefore $a \in \mathrm{supp}^+(\xi_a - \xi_b)$ and $b \in \mathrm{supp}^-(\xi_a - \xi_b)$. We make use of the conformal decomposition $\xi_a - \xi_b = \sum_{i=1}^{m}\beta_i\pi_i$, where we assume $\pi_i(a) = 1$ for $i = 1, 2, \ldots, \ell$ and $\pi_i(a) = 0$ for $i = \ell+1, \ell+2, \ldots, m$. We have $\pi_i(b) = 0$ for $i = 1, 2, \ldots, \ell$ by Proposition 3.1 (ii), since S is a set of complements and $a \in \mathrm{supp}^+(\xi_a - \xi_b)$ and $b \in \mathrm{supp}^-(\xi_a - \xi_b)$. Then $\xi = \xi_a - \sum_{i=1}^{\ell}\beta_i\pi_i$ and $\widetilde{\xi} = \xi_b + \sum_{i=1}^{\ell}\beta_i\pi_i$ satisfy (3.16).

To show (3.15) let ξ_a and ξ_S be optimal solutions for $c+\lambda\chi_a$ and $c-\mu\chi_S$. We can establish (3.15) by constructing vectors ξ and $\widetilde{\xi}$ with $N\xi = N\widetilde{\xi} = \mathbf{0}$ such that

$$\left.\begin{array}{l} \xi + \widetilde{\xi} = \xi_a + \xi_S, \\ d \leq \xi \leq c, \quad d \leq \widetilde{\xi} \leq c + \lambda\chi_a - \mu\chi_S, \\ \xi_a \wedge \xi_S \leq \xi \wedge \widetilde{\xi} \leq \xi \vee \widetilde{\xi} \leq \xi_a \vee \xi_S. \end{array}\right\} \tag{3.17}$$

If $\xi_a(a) \leq c(a)$, we can take $\xi = \xi_a$ and $\widetilde{\xi} = \xi_S$ to meet (3.17). Otherwise, we have $\xi_a(a) > c(a) \geq \xi_S(a)$, and therefore $a \in \mathrm{supp}^+(\xi_a - \xi_S)$. We use the conformal decomposition $\xi_a - \xi_S = \sum_{i=1}^m \beta_i \pi_i$. Since S is a set of complements we may assume by Proposition 3.4 that

$$a \in \mathrm{supp}^+(\pi_1) \cap S \subseteq \mathrm{supp}^+(\pi_2) \cap S \subseteq \cdots \subseteq \mathrm{supp}^+(\pi_\ell) \cap S$$

and $\pi_i(a) = 0$ for $i = \ell+1, \ell+2, \ldots, m$; then $\mathrm{supp}^-(\pi_i) \cap S = \emptyset$ for $i = 1, 2, \ldots, \ell$. Noting $\sum_{i=1}^{\ell} \beta_i = \xi_a(a) - \xi_S(a) \geq \xi_a(a) - c(a)$, let k be the smallest integer with $\sum_{i=1}^{k} \beta_i \geq \xi_a(a) - c(a)$ and define $\beta' = [\xi_a(a) - c(a)] - \sum_{i=1}^{k-1} \beta_i$. Then $\xi = \xi_a - \sum_{i=1}^{k-1} \beta_i\pi_i - \beta'\pi_k$ and $\widetilde{\xi} = \xi_S + \sum_{i=1}^{k-1} \beta_i\pi_i + \beta'\pi_k$ satisfy (3.17), since

$$\begin{aligned} \xi(a) &= \xi_a(a) - \sum_{i=1}^{k-1} \beta_i - \beta' = c(a), \\ \widetilde{\xi}(a) &= \xi_S(a) + \sum_{i=1}^{k-1} \beta_i + \beta' \ = \ \xi_S(a) + \xi_a(a) - c(a) \ \leq \ c(a) + \lambda - \mu, \end{aligned}$$

and, for any $b \in \mathrm{supp}^+(\pi_k) \cap S \setminus \{a\}$, we have

$$\begin{aligned} \widetilde{\xi}(b) &= \xi_a(b) + [\beta' - \sum_{i=k}^{l} \beta_i - \sum_{i=l+1}^{m} \beta_i\pi_i(b)] \ \leq \ c(b) + [\beta' - \sum_{i=k}^{l} \beta_i] \\ &= c(b) + [\xi_S(a) - c(a)] \ \leq \ c(b) + [(c(a) - \mu) - c(a)] \ \leq \ c(b) - \mu. \end{aligned}$$

Acknowledgement

The authors thank anonymous referees for their valuable comments on the manuscript. This work is supported by Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

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