Neighbor Systems, Jump Systems, and Bisubmodular Polyhedra[1]

Akiyoshi SHIOURA[2]

Abstract

The concept of neighbor system, introduced by Hartvigsen (2010), is a set of integral vectors satisfying a certain combinatorial property. In this paper, we reveal the relationship of neighbor systems with jump systems and with bisubmodular polyhedra. We firstly prove that for every neighbor system, there exists a jump system which has the same neighborhood structure as the original neighbor system. This shows that the concept of neighbor system is essentially equivalent to that of jump system. We next show that the convex closure of a neighbor system is an integral bisubmodular polyhedron. In addition, we give a characterization of neighbor systems using bisubmodular polyhedra. Finally, we consider the problem of minimizing a separable convex function on a neighbor system. It is shown that the problem can be solved in weakly-polynomial time for a class of neighbor systems.

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[2]Graduate School of Information Sciences, Tohoku University, Sendai 980-8579, Japan, shioura@dais.is.tohoku.ac.jp

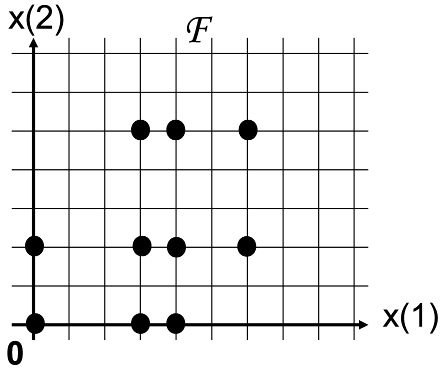

Figure 1: An example of 2-dimensional neighbor system $\mathcal{F}$, where the black dots represents integral vectors in the neighbor system

1 Introduction

The concept of neighbor system, introduced by Hartvigsen [14], is a set of integral vectors satisfying a certain combinatorial property. The definition of neighbor system is as follows. Throughout this paper, let E be a finite set with n elements. Let $\mathcal{F}$ be a set of integral vectors in $\mathbb{Z}^E$. For $x, y \in \mathcal{F}$, we say that y is a *neighbor* of x if there exist some vector $d \in \{0, +1, -1\}^E$ with exactly one or two nonzero components and a positive integer α such that

$$y = x + \alpha d \text{ and } x + \alpha' d \notin \mathcal{F} \text{ for all } \alpha' \in \mathbb{Z} \text{ with } 0 < \alpha' < \alpha.$$

For vectors $x, y, z \in \mathbb{Z}^E$, z is said to be *between x and y* if the following inequality holds:

$$\min\{x(e), y(e)\} \leq z(e) \leq \max\{x(e), y(e)\} \quad (\forall e \in E).$$

The set $\mathcal{F}$ is called an *(all-)neighbor system* if it satisfies the following axiom:

> for every $x, y \in \mathcal{F}$ and $e \in E$ with $x(e) \neq y(e)$, there exists a neighbor $z \in \mathcal{F}$ of x such that $z(e) \neq x(e)$ and z is between x and y.

See Figure 1 for an example of 2-dimensional neighbor system. Given a positive integer k, a neighbor system $\mathcal{F}$ is said to be an *N_k-neighbor system* if we can always choose a neighbor z in the axiom above such that $||z - x||_1 \leq k$. For example, the neighbor system in Figure 1 is an N_k-neighbor system for every $k \geq 3$, but not for $k = 1, 2$ since if $x = (0, 2)$ and $y = (3, 5)$ then we do not have a neighbor z of x such that z is between x and y and satisfies $||z - x||_1 \leq 2$.

Neighbor system is a common generalization of various concepts such as matroid, integral polymatroid, delta-matroid, integral bisubmodular polyhedron, and jump systems. Below we review these concepts.

Matroids. The concept of matroid is introduced by Whitney [24]. One of the important results on matroids, from the viewpoint of combinatorial optimization, is the validity of a greedy algorithm for linear optimization (see, e.g., [11]).

Integral Polymatroids. The concept of polymatroid is introduced by Edmonds [10] as a generalization of matroids. A polymatroid is a polyhedron defined by a monotone submodular function, and a greedy algorithm for matroids can be naturally extended to polymatroids. The minimization of separable convex function can be also done in a greedy way, and efficient algorithms have been proposed (see, e.g., [13, 15]). An integral polymatroid is a polymatroid which is an integral polyhedron, i.e., all extreme points are given by integer vectors.

Delta-Matroids. The concept of delta-matroid (or pseudomatroid) is introduced by Bouchet [5] and Chandrasekaran and Kabadi [7]. A delta-matroid can be seen as a family of subsets of a ground set with a nice combinatorial structure, and generalizes the concept of matroid. A more general greedy algorithm works for the linear optimization on a delta-matroid.

Integral Bisubmodular Polyhedron. The concept of bisubmodular polyhedron (or polypseudomatroid), introduced by Dunstan and Welsh [9] (see also [6, 7, 12]), is a common generalization of polymatroid and delta-matroid. The concept of bisubmodular polyhedron is defined by using bisubmodular functions. A function $\rho : 3^E \to \mathbb{R} \cup \{+\infty\}$ is said to be *bisubmodular* if it satisfies the bisubmodular inequality, where $3^E = \{(X, Y) \mid X, Y \subseteq E,\ X \cap Y = \emptyset\}$:

$$\begin{aligned} &\rho(X_1, Y_1) + \rho(X_2, Y_2) \\ &\quad \geq \ \rho((X_1 \cup X_2) \setminus (Y_1 \cup Y_2), (Y_1 \cup Y_2) \setminus (X_1 \cup X_2)) + \rho(X_1 \cap X_2, Y_1 \cap Y_2) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\forall (X_1, Y_1), (X_2, Y_2) \in 3^E). \end{aligned}$$

For a function $\rho : 3^E \to \mathbb{R} \cup \{+\infty\}$ with $\rho(\emptyset, \emptyset) = 0$, we define a polyhedron $\mathrm{P}_*(\rho) \subseteq \mathbb{R}^E$ by

$$\mathrm{P}_*(\rho) = \{x \in \mathbb{R}^E \mid \sum_{i \in X} x(i) - \sum_{i \in Y} x(i) \leq \rho(X, Y) \ ((X, Y) \in 3^E)\},$$

which is called a *bisubmodular polyhedron* if ρ is bisubmodular. Bisubmodular polyhedra constitute a class of polyhedra on which a simple greedy algorithm works for linear optimization. In addition, separable convex function minimization can be done in a greedy manner [2]. In this paper, we are mainly interested in integral bisubmodular polyhedra; a polyhedron $P \subseteq \mathbb{R}^E$ is said to be *integral* if the convex closure of $P \cap \mathbb{Z}^E$ is equal to P.

Jump Systems. The concept of jump system is introduced by Bouchet and Cunningham [6], which is a common generalization of delta-matroid and the set of integral vectors in an integral bisubmodular polyhedron. We give a precise definition of jump systems. For $e \in E$, the characteristic vector $\chi_e \in \{0, 1\}^E$ is the vector such that $\chi_e(i) = 1$ and $\chi_e(i) = 0$ for $i \in E \setminus \{e\}$. Denote by U the set of vectors $+\chi_e, -\chi_e$ $(e \in E)$. For vectors $x, y \in \mathbb{Z}^E$, define

$$\mathrm{inc}(x, y) = \{p \in U \mid x + p \text{ is between } x \text{ and } y\}.$$

A set $\mathcal{J} \subseteq \mathbb{Z}^E$ is a *jump system* if it satisfies the following axiom:

(J) for every $x, y \in \mathcal{J}$ and every $p \in \mathrm{inc}(x, y)$, if $x + p \notin \mathcal{J}$ then there exists $q \in \mathrm{inc}(x + p, y)$ such that $x + p + q \in \mathcal{J}$.

Interesting examples of jump systems can be found in the set of degree sequences of the subgraphs of undirected and directed graphs; for example, matchings and b-matchings in undirected graphs [6, 8, 17] and even-factors in directed graphs [18]. Validity of certain greedy algorithms is shown in [6] for the linear optimization and in [3] for separable convex function minimization. Moreover, a polynomial-time algorithm for separable convex function minimization is given in [23]. It is shown that a jump system is equivalent to an N_2-neighbor system [14].

We give two examples of neighbor systems which are not jump systems; such an example is also given in Figure 1.

Example 1.1 (Expansion of jump systems)**.** For a jump system $\mathcal{J} \subseteq \mathbb{Z}^E$ and a positive integer k, the set $\{kx \in \mathbb{Z}^E \mid x \in \mathcal{J}\}$ is an N_{2k}-neighbor system [14].

Example 1.2 (Rectilinear grid)**.** Let $u \in \mathbb{Z}^E_+$ be a nonnegative vector, and for $e \in E$, let $\pi_e : [0, u(e)] \to \mathbb{Z}$ be a strictly increasing function. Then, the set of $(\pi_e(x(e)) \mid e \in E)$ for vectors $x \in \mathbb{Z}^E_+$ with $x \leq u$ is an all-neighbor system.

These examples, in particular, show that a neighbor system may have a "hole," as in the case of jump system, and it can be arbitrarily large.

Neighbor systems provide a systematic and simple way to characterize matroids and its generalizations for which greedy algorithms work for linear optimization. Indeed, it is shown that linear optimization on a neighbor system can be solved by a greedy algorithm, and that the greedy algorithm runs in polynomial time for finite N_k-neighbor systems [14].

Our Results. The main aim of this paper is to reveal the relationship of neighbor systems with jump systems and with bisubmodular polyhedra.

We firstly prove in Section 3 that for every neighbor system $\mathcal{F} \subseteq \mathbb{Z}^E$, there exists a jump system $\mathcal{J} \subseteq \mathbb{Z}^E$ which has the same neighborhood structure as $\mathcal{F}$ (see Theorem 3.1). This means that the concept of neighbor system is essentially equivalent to that of jump system, although the class of neighbor systems properly contains that of jump systems. Our result implies that every property of jump systems can be restated in terms of neighbor systems by using the equivalence.

We then discuss the relationship between neighbor systems and bisubmodular polyhedra in Section 4. It is known that the convex closure of a jump system, which is a special case of neighbor systems, is an integral bisubmodular polyhedron [6]. We show that the convex closure of a neighbor system is also an integral bisubmodular polyhedron (see Theorem 4.2). In addition, we give a characterization of neighbor systems using bisubmodular polyhedra, stating that a set of integral vectors is a neighbor system if and only if the convex closure of its restriction with an interval is always an integral bisubmodular polyhedron (see Theorem 4.3).

In Section 5, we consider a linear optimization on a neighbor system. We show that the results in Section 4 implies that a simple greedy algorithm for the linear optimization on a bisubmodular polyhedron can be also used for neighbor systems. We also discuss the relationship between this greedy algorithm and an algorithm proposed by Hartvigsen [14].

We consider the separable convex optimization problem on neighbor systems in Section 6. Given a family of univariate convex functions $f_e : \mathbb{Z} \to \mathbb{R}$ $(e \in E)$ and a finite neighbor system $\mathcal{F} \subseteq \mathbb{Z}^E$, we consider the following problem:

$$\textbf{(SC)}\ \text{Minimize} \sum_{e \in E} f_e(x(e)) \text{ subject to } x \in \mathcal{F}.$$

For a special case where $\mathcal{F}$ is a jump system, it is shown that the problem (SC) can be solved in pseudo-polynomial time by a greedy-type algorithm [3], and in weakly-polynomial time by an algorithm called the domain reduction algorithm [23]. We extend these algorithms for jump systems to neighbor systems.

Organization of This Paper. Section 2 is devoted to preliminaries on the fundamental concepts discussed in this paper. We discuss the relationship of neighbor systems with jump systems and with bisubmodular polyhedra in Sections 3 and 4, respectively. In Sections 5 and 6, we consider linear and separable convex optimization problems, respectively, and present efficient algorithms.

2 Preliminaries

Throughout this paper, let E be a finite set with n elements. We denote by $\mathbb{Z}, \mathbb{Z}_+, \mathbb{Z}_{++}$ the sets of integers, nonnegative integers, and positive integers, respectively. We denote by $\mathbb{R}$ the set of real numbers. For vectors $\ell \in (\mathbb{Z} \cup \{-\infty\})^E$ and $u \in (\mathbb{Z} \cup \{+\infty\})^E$ with $\ell \leq u$, we define the integer interval $[\ell, u]$ as the set of integral vectors $x \in \mathbb{Z}^E$ with $\ell(e) \leq x(e) \leq u(e)$ $(\forall e \in E)$. For a vector $x \in \mathbb{R}^E$, we define $\mathrm{supp}(x) = \{e \in E \mid x(e) \neq 0\}$. For $e \in E$, the characteristic vector $\chi_e \in \{0,1\}^E$ is the vector such that $\chi_e(i) = 1$ and $\chi_e(i) = 0$ for $i \in E \setminus \{e\}$. We denote by $\mathbf{0}$ the zero vector in $\mathbb{Z}^E$.

We review the original definition of neighbor systems in [14] using the concept of neighbor function. We define $U \subseteq \mathbb{Z}^E$ to be the set of unit vectors, i.e.,

$$U = \{x \in \mathbb{Z}^E \mid ||x||_1 = 1\} = \{+\chi_e \mid e \in E\} \cup \{-\chi_e \mid e \in E\}. \tag{2.1}$$

A *direction* is a $\{0, +1, -1\}$-vector with exactly one or two nonzero components, and denote by D $(\subseteq \{0, +1, -1\}^E)$ the set of all directions. That is, D is given as

$$\begin{aligned} D \;=\; & U \cup \{+\chi_e + \chi_{e'} \mid e, e' \in E,\ e \neq e'\} \cup \{-\chi_e - \chi_{e'} \mid e, e' \in E,\ e \neq e'\} \\ & \cup \{+\chi_e - \chi_{e'} \mid e, e' \in E,\ e \neq e'\}. \end{aligned}$$

Let $\mathcal{F}$ be a set of integral vectors in $\mathbb{Z}^E$. For $x, y \in \mathcal{F}$, we say that y is a *neighbor* of x if there exist some direction $d \in D$ and a positive integer α such that

$$y = x + \alpha d \text{ and } x + \alpha' d \notin \mathcal{F} \text{ for all } \alpha' \in \mathbb{Z} \text{ with } 0 < \alpha' < \alpha.$$

A *neighbor function*, denoted by N, is a function that takes as input any set $\mathcal{F} \subseteq \mathbb{Z}^E$ with any $x \in \mathcal{F}$ and outputs a subset of the neighbors of x in $\mathcal{F}$, denoted by $N(\mathcal{F}, x)$. In particular, $N^a(\mathcal{F}, x)$ (resp., $N_k(\mathcal{F}, x)$) denotes the set of all neighbors of x in $\mathcal{F}$ (resp., the set of all neighbors y of x in $\mathcal{F}$ with $||y - x||_1 \leq k$). For vectors $x, y, z \in \mathbb{Z}^E$, z is said to be *between x and y* if the following inequality holds:

$$\min\{x(e), y(e)\} \leq z(e) \leq \max\{x(e), y(e)\} \quad (\forall e \in E).$$

Given a set $\mathcal{F} \subseteq \mathbb{Z}^E$ and a neighbor function N, we say that $\mathcal{F}$ is an *N-neighbor system* if the following condition holds:

(NS) for every $x, y \in \mathcal{F}$ and every $e \in \mathrm{supp}(x - y)$, there exists $z \in N(\mathcal{F}, x)$ such that z is between x and y and $z(e) \neq x(e)$.

An N-neighbor system is an *all-neighbor system* if $N = N^a$, and an N_k*-neighbor system* if $N = N_k$.

In the following discussion, we use an equivalent axiom of neighbor systems given below. Recall that U is the set of unit vectors; see (2.1). For $x, y \in \mathbb{Z}^E$ the set $\mathrm{inc}(x, y)$ is defined as

$$\mathrm{inc}(x, y) = \{p \in U \mid x + p \text{ is between } x \text{ and } y\}.$$

Then, (NS) can be rewritten as follows.

(NS′) for every $x, y \in \mathcal{F}$ and every $p \in \mathrm{inc}(x, y)$, there exist $q \in \mathrm{inc}(x, y) \cup \{\mathbf{0}\} \setminus \{+p, -p\}$ and $\alpha \in \mathbb{Z}_{++}$ such that $x' \equiv x + \alpha(p + q) \in N(\mathcal{F}, x)$ and x' is between x and y.

We note that the axiom (NS′) is similar to the axiom of jump systems; a nonempty set $\mathcal{J} \subseteq \mathbb{Z}^E$ is a *jump system* if it satisfies the following axiom:

(J) for every $x, y \in \mathcal{J}$ and every $p \in \mathrm{inc}(x, y)$, if $x + p \notin \mathcal{J}$ then there exists $q \in \mathrm{inc}(x + p, y)$ such that $x + p + q \in \mathcal{J}$.

The class of neighbor systems is closed under the following operations.

Proposition 2.1 (cf. [14])**.** *Let $\mathcal{F} \subseteq \mathbb{Z}^n$ be an N-neighbor system.*
(i) *For a positive integer $m > 0$, define a set $\mathcal{F}'$ and a neighbor function N' by*

$$\mathcal{F}' = \{mx \mid x \in \mathcal{F}\}, \qquad N'(\mathcal{F}', mx) = \{my \mid y \in N(\mathcal{F}, x)\}.$$

Then, $\mathcal{F}'$ is an N'-neighbor system.
(ii) *For a vector $s \in \{+1, -1\}^E$, we define a set $\mathcal{F}_s$ and a neighbor function N_s by*

$$\begin{aligned} \mathcal{F}_s &= \{(s(e)x(e) \mid e \in E) \mid x \in \mathcal{F}\}, \\ N_s(\mathcal{F}_s, y) &= \{(s(e)x'(e) \mid e \in E) \mid x' \in N(\mathcal{F}, x)\} \text{ for } y = (s(e)x(e) \mid e \in E) \in \mathcal{F}_s. \end{aligned}$$

Then, $\mathcal{F}_s$ is an N_s-neighbor system.
(iii) *For vectors $\ell, u \in \mathbb{Z}^E$ with $\ell \leq u$ and $\mathcal{F} \cap [\ell, u] \neq \emptyset$, the set $\mathcal{F} \cap [\ell, u]$ is an N-neighbor system.*
(iv) *For a vector $a \in \mathbb{Z}^E$, the set $\mathcal{F} + a \equiv \{x + a \mid x \in \mathcal{F}\}$ is an N'-neighbor system, where $N'(\mathcal{F} + a, x) = \{y + a \mid y \in N(\mathcal{F}, x)\}$.*

We introduce a concept of *proper neighbor*, which is a neighbor satisfying an additional condition. For a neighbor system $\mathcal{F}$ and vectors $x, y \in \mathcal{F}$, we say that y is a *proper neighbor of x in $\mathcal{F}$* if y is a neighbor of x satisfying either of the conditions (i) or (ii), where

(i) there exist some $\alpha \in \mathbb{Z}_{++}$ and $p \in U$ such that $y - x = \alpha p$,
(ii) there exist some $\alpha \in \mathbb{Z}_{++}$ and $p, q \in U$ with $p \notin \{q, -q\}$ such that $y - x = \alpha(p + q)$ and $x + \alpha' p \notin \mathcal{F}$ for all $\alpha' \in \mathbb{Z}$ with $0 < \alpha' \leq \alpha$.

Note that if (i) holds, then we have $x + \alpha' p \notin \mathcal{F}$ for all $\alpha' \in \mathbb{Z}$ with $0 < \alpha' \leq \alpha$ since y is a neighbor of x.

To illustrate the concept of proper neighbor, consider the neighbor system in Figure 1. The vector $y = (6, 2)$ is a proper neighbor of $x = (4, 0)$ since $(5, 0)$ and $(6, 0)$ are not in $\mathcal{F}$ and therefore the condition (ii) holds with $p = (1, 0)$, $q = (0, 1)$, and $\alpha = 2$. The vector $(6, 5)$ is a neighbor of $(3, 2)$, but not a proper neighbor of $(3, 2)$ since $(4, 2), (3, 5) \in \mathcal{F}$.

The next property shows that the concept of proper neighbor is essential in the definition of neighbor systems.

Theorem 2.2. *Let $\mathcal{F} \subseteq \mathbb{Z}^E$.*
(i) *$\mathcal{F}$ is an all-neighbor system if and only if for every $x, y \in \mathcal{F}$ and every $e \in \mathrm{supp}(x - y)$, there exists a proper neighbor $z \in \mathcal{F}$ of x such that z is between x and y and $z(e) \neq x(e)$.*
(ii) *$\mathcal{F}$ is an N_k-neighbor system for some $k \in \mathbb{Z}_{++}$ if and only if for every $x, y \in \mathcal{F}$ and every $e \in \mathrm{supp}(x - y)$, there exists a proper neighbor $z \in N_k(\mathcal{F}, x)$ such that z is between x and y and $z(e) \neq x(e)$.*

Proof. Since the "if" part of (i) and (ii) is obvious from the definition of neighbor systems, we show the "only if" part of (i) and (ii).

[Proof of the "only if" part for (i)] Let $x, y \in \mathcal{F}$ and $e \in \mathrm{supp}(x - y)$. By the property (NS), there exists a neighbor $z \in \mathcal{F}$ of x such that z is between x and y and $z(e) \neq x(e)$. Suppose that z is not a proper neighbor of x. Then, we have $|\mathrm{supp}(z - x)| = 2$ by the definition of proper neighbor. We may assume, without loss of generality, that $z = x + \alpha(\chi_e + \chi_i)$ for some $i \in E \setminus \{e\}$ and $\alpha \in \mathbb{Z}_{++}$. Since z is not a proper neighbor of x, there exist some $\alpha' \in \mathbb{Z}$ such that $0 < \alpha' \leq \alpha$ and $x + \alpha' \chi_e \in \mathcal{F}$, which is a proper neighbor of x.

[Proof of the "only if" part for (ii)] Let $x, y \in \mathcal{F}$ and $e \in \mathrm{supp}(x - y)$. From the statement (i) we see that there exists a proper neighbor $z \in \mathcal{F}$ of x such that z is between x and y and $z(e) \neq x(e)$. Hence, it suffices to proven that $\|z - x\|_1 \leq k$ holds, which is shown in Proposition 2.3 below. $\square$

Proposition 2.3. *Let $\mathcal{F} \subseteq \mathbb{Z}^E$ be an N_k-neighbor system with some $k \geq 1$, and $x \in \mathcal{F}$. Then, every proper neighbor y of x is contained in $N_k(\mathcal{F}, x)$.*

Proof. Let y be a proper neighbor of x, and assume, to the contrary, that $\|y - x\|_1 > k$.

Suppose that $|\mathrm{supp}(y - x)| = 1$. By Proposition 2.1 (ii), we may assume, without loss of generality, that $y = x + \alpha \chi_e$ for some $e \in E$ and $\alpha \in \mathbb{Z}$ with $\alpha > k$. Since $\mathcal{F}$ is an N_k-neighbor system, the property (NS) applied to x and y implies that $x + \beta \chi_e \in \mathcal{F}$ for some $\beta \in \mathbb{Z}$ with $0 < \beta \leq k$, a contradiction to the fact that y is a neighbor of x.

We then suppose that $|\mathrm{supp}(y - x)| = 2$. We may assume, without loss of generality, that $y = x + \alpha(\chi_i + \chi_j)$ for some distinct $i, j \in E$ and $\alpha \in \mathbb{Z}$ with $\alpha > k$. By applying (NS) to x, y, and $i \in \mathrm{supp}(x - y)$, we obtain some $z \in N_k(\mathcal{F}, x)$ such that $z(i) > x(i)$ and z is between x and y. Since $\|z - x\|_1 \leq k$, we have $z \neq y$, a contradiction to the assumption that y is a proper neighbor of x. $\square$

Remark 2.4. Let $\mathcal{F} \subseteq \mathbb{Z}^E$ be an N_k-neighbor system. Proposition 2.3 shows that every proper neighbor y of x is contained in $N_k(\mathcal{F}, x)$. On the other hand, it is possible that there exists a neighbor y of $x \in \mathcal{F}$ such that $y \notin N_k(\mathcal{F}, x)$. For example, consider the set $\mathcal{F} \subseteq \mathbb{Z}^2$ given in

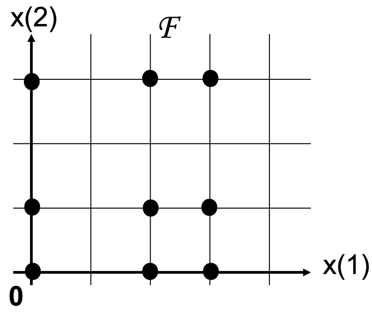

Figure 2: An example of 2-dimensional N_2-neighbor system

Figure 2, which is an N_2-neighbor system. For $x = (0, 0)$, the vector $y = (3, 3)$ is a neighbor of x since $(1, 1), (2, 2) \notin \mathcal{F}$, and it holds that $y \notin N_2(\mathcal{F}, x) = \{(2, 0), (0, 1)\}$. Note that y is not a proper neighbor of x. $\square$

For a jump system, which is a special case of neighbor system, the conditions defining (proper) neighbors can be simplified as follows.

Proposition 2.5. *Let $\mathcal{J} \subseteq \mathbb{Z}^E$ be a jump system $\mathcal{J}$ and $x, y \in \mathcal{J}$.*
(i) *y is a proper neighbor of x in $\mathcal{J}$ if and only if either of the following conditions holds:*

(a) *there exists some $p \in U$ such that $y - x = p$, or $y - x = 2p$ and $x + p \notin \mathcal{J}$,*
(b) *there exist some $p, q \in U$ with $p \notin \{+q, -q\}$ such that $y - x = p + q$ and $x + p \notin \mathcal{J}$.*

(ii) *y is a neighbor of x in $\mathcal{J}$ if and only if either of the conditions* (a) *and* (b′) *below holds:*

(b′) *there exist some $p, q \in U$ with $p \notin \{+q, -q\}$ such that $y - x = p + q$.*

Proof. We prove (i) only. It is easy to see from the definition of proper neighbor that if either of the conditions (a) and (b) holds, then y is a proper neighbor of x in $\mathcal{J}$. Hence, it suffices to show the "only if" part of the statement. Let $x, y \in \mathcal{J}$ be vectors such that y is a proper neighbor of x in $\mathcal{J}$. We will show that either (a) or (b) holds.

We firstly assume that $y - x = \alpha p$ holds for some $\alpha \in \mathbb{Z}_{++}$ and $p \in U$. If $\alpha = 1$, then (a) holds immediately. Otherwise (i.e., $\alpha > 1$), we have $x + p \notin \mathcal{J}$, and the axiom (J) for jump systems applied to x and y implies that $y = x + 2p \in \mathcal{J}$, i.e., (a) holds.

We next assume that there exist some $\alpha \in \mathbb{Z}_{++}$ and $p, q \in U$ with $p \notin \{+q, -q\}$ such that $y - x = \alpha(p + q)$ and $x + \alpha' p \notin \mathcal{F}$ for all $\alpha' \in \mathbb{Z}$ with $0 < \alpha' \leq \alpha$. If $\alpha = 1$, then (b) holds immediately. Otherwise, we have $x + p \notin \mathcal{J}$, $x + 2p \notin \mathcal{J}$, and the axiom (J) for jump systems applied to x and y implies $x + p + q \in \mathcal{J}$, i.e., (b) holds. $\square$

Remark 2.6. The conditions (a) and (b) in Proposition 2.5 are considered in [3] to define a local optimality for the separable convex minimization problem on jump systems. $\square$

3 Relationship between Neighbor Systems and Jump Systems

3.1 Result

We discuss the relationship between neighbor systems and jump systems. It is shown that for every neighbor system, there exists a jump system which has the same neighborhood structure as the original neighbor system.

Theorem 3.1. *Let $\mathcal{F} \subseteq \mathbb{Z}^E$ be an all-neighbor system. Then, there exist a jump system $\mathcal{J} \subseteq \mathbb{Z}^E$ and a bijective function $\pi : \mathcal{J} \to \mathcal{F}$ satisfying the following property, where $\pi^{-1} : \mathcal{F} \to \mathcal{J}$ is the inverse function of π:*

> *for every $x, y \in \mathcal{F}$, the vector x is a proper neighbor of y in $\mathcal{F}$ if and only if $\pi^{-1}(x)$ is a proper neighbor of $\pi^{-1}(y)$ in $\mathcal{J}$.*

We give a proof of Theorem 3.1. Let $\mathcal{F} \subseteq \mathbb{Z}^E$ be an all-neighbor system. By Proposition 2.1 (iv), we may assume, without loss of generality, that $\mathcal{F}$ contains the zero vector $\mathbf{0}$. For $e \in E$, we define a set $\mathcal{F}_e \subseteq \mathbb{Z}$ by

$$\mathcal{F}_e = \{\alpha \mid \alpha \in \mathbb{Z},\ \exists x \in \mathcal{F} \text{ s.t. } x(e) = \alpha\}.$$

Define the numbers $u(e) \in \mathbb{Z} \cup \{+\infty\}$ and $\ell(e) \in \mathbb{Z} \cup \{-\infty\}$ by

$$\begin{aligned} u(e) &= \text{the number of positive integers in } \mathcal{F}_e, \\ l(e) &= -(\text{the number of negative integers in } \mathcal{F}_e). \end{aligned}$$

We also define a function $\pi_e : [\ell(e), u(e)] \to \mathbb{Z}$ by $\pi_e(0) = 0$ and

$$\begin{aligned} \pi_e(k) &= \text{ the } k\text{-th smallest positive integer in } \mathcal{F}_e \qquad (\text{if } 0 < k \le u(e)), \\ \pi_e(-k) &= \text{ the } k\text{-th largest negative integer in } \mathcal{F}_e \qquad (\text{if } \ell(e) \le -k < 0). \end{aligned}$$

Then, each π_e is a strictly increasing function in the interval $[\ell(e), u(e)]$, and gives a bijection between $[\ell(e), u(e)]$ and $\mathcal{F}_e$. We define a set $\mathcal{J} \subseteq \mathbb{Z}^E$ and a function $\pi : \mathcal{J} \to \mathcal{F}$ by

$$\mathcal{J} = \{z \in \mathbb{Z}^E \mid (\pi_e(z(e)) \mid e \in E) \in \mathcal{F}\}, \qquad \pi(z) = (\pi_e(z(e)) \mid e \in E) \quad (z \in \mathcal{J}).$$

By the definitions of π_e and $\mathcal{J}$, the function π is bijective.

To complete the proof of Theorem 3.1, it suffices to show the following:

Lemma 3.2.
(i) *The set $\mathcal{J}$ is a jump system.*
(ii) *For every $x, y \in \mathcal{F}$, the vector x is a proper neighbor of y in $\mathcal{F}$ if and only if $\pi^{-1}(x)$ is a proper neighbor of $\pi^{-1}(y)$ in $\mathcal{J}$.*

The proof is given in Section 3.2.

For illustration of the relationship between neighbor systems and jump systems, we again consider the neighbor system given in Figure 1 (see Figure 3). In this example, we have $E = \{1, 2\}$, $\mathcal{F}_1 = \{0, 3, 4, 6\}$, $\mathcal{F}_2 = \{0, 2, 5\}$, $u(1) = 3$, $l(1) = 0$, $u(2) = 2$, and $l(2) = 0$. Hence, the functions $\pi_1 : [0, 3] \to \mathbb{Z}$ and $\pi_2 : [0, 2] \to \mathbb{Z}$ are given by

$$\pi_1(0) = 0,\ \pi_1(1) = 3,\ \pi_1(2) = 4,\ \pi_1(3) = 6, \quad \pi_2(0) = 0,\ \pi_2(1) = 2,\ \pi_2(2) = 5.$$

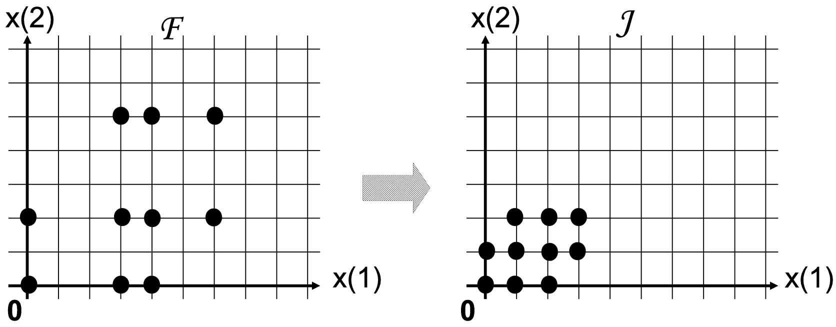

Figure 3: Relationship between neighbor systems and jump systems

The resulting set $\mathcal{J}$ is shown in Figure 3, which is a jump system. It is easy to see that there exists a one-to-one correspondence between points in the neighbor system $\mathcal{F}$ and points in the jump system $\mathcal{J}$.

Theorem 3.1 implies the following technical lemma, which will be used in Section 6.1.

Lemma 3.3. *Let $x \in \mathcal{F}$, $i, j \in E$ be distinct elements, and $\alpha \in \mathbb{Z}_{++}$. Suppose that $x' \equiv x + \alpha(\chi_i + \chi_j)$ is a proper neighbor of x. Then,*

$$x'(i) = \min\{y(i) \mid y \in \mathcal{F},\ y(i) > x(i)\}, \qquad x'(j) = \min\{y(j) \mid y \in \mathcal{F},\ y(j) > x(j)\}.$$

Proof. Consider the jump system $\mathcal{J}$ and the bijective function $\pi : \mathcal{J} \to \mathcal{F}$ in the proof of Theorem 3.1 above. Then, it holds that $\pi^{-1}(x + \alpha(\chi_i + \chi_j)) = \pi^{-1}(x) + \chi_i + \chi_j$. By the definition of the function π, we have

$$\{y \in \mathcal{F} \mid x(i) < y(i) < x'(i)\} = \emptyset, \qquad \{y \in \mathcal{F} \mid x(j) < y(j) < x'(j)\} = \emptyset,$$

which implies the statement of the lemma. □

3.2 Proof of Lemma 3.2

3.2.1 Proof of Lemma 3.2 (i)

To prove Lemma 3.2 (i), we use the following lemmas. Recall that $U = \{+\chi_e \mid e \in E\} \cup \{-\chi_e \mid e \in E\}$ (see (2.1)).

Lemma 3.4. *Let $z \in \mathcal{F}$ and suppose that $z + \alpha p + \beta q \in \mathcal{F}$ holds for some $p, q \in U$ with $p \notin \{+q, -q\}$ and $\alpha, \beta \in \mathbb{Z}$ with $\alpha \geq \beta \geq 0$. Then, there exists some $\varepsilon \in \mathbb{Z}$ with $0 \leq \varepsilon \leq \alpha$ such that $z + (\alpha - \varepsilon)p \in \mathcal{F}$.*

Proof. The proof is done by induction on β. If $\beta = 0$, then the claim holds with $\varepsilon = 0$. Hence, we assume $\beta \geq 1$. By the property (NS) applied to $z + \alpha p + \beta q$ and z, there exist some $\delta \in \mathbb{Z}$ with $0 < \delta \leq \beta$ and some $\delta' \in \{0, \delta\}$ such that $z + (\alpha - \delta')p + (\beta - \delta)q$ is a vector in $\mathcal{F}$ between $z + \alpha p + \beta q$ and z. Note that $0 \leq \beta - \delta \leq \alpha - \delta'$ holds. By the induction hypothesis applied to $z + (\alpha - \delta')p + (\beta - \delta)q$, there exists some $\varepsilon' \in \mathbb{Z}$ with $0 \leq \varepsilon' \leq \beta - \delta$ such that $z + (\alpha - \delta' - \varepsilon')p \in \mathcal{F}$. Since $\delta' + \varepsilon' \leq \beta$, the claim holds with $\varepsilon = \delta' + \varepsilon'$. □

Lemma 3.5. *Let $z \in \mathcal{F}$, and $p_1, p_2, p_3 \in U$ be vectors such that $p_h \notin \{p_k, -p_k\}$ for all distinct $h, k \in \{1,2,3\}$. Suppose that $z + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 \in \mathcal{F}$ holds for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Z}_+$ with $\alpha_1 + \alpha_2 \geq \alpha_3$. Then, there exist some integers ε_1 and ε_2 such that $0 \leq \varepsilon_1 \leq \alpha_1$, $0 \leq \varepsilon_2 \leq \alpha_2$, $\varepsilon_1 + \varepsilon_2 \leq \alpha_3$, and $z + (\alpha_1 - \varepsilon_1)p_1 + (\alpha_2 - \varepsilon_2)p_2 \in \mathcal{F}$.*

Proof. The proof is similar to that for Lemma 3.4 and therefore omitted. ☐

We now show that $\mathcal{J}$ is a jump system. Let $\tilde{x}, \tilde{y} \in \mathcal{J}$, and $p \in \mathrm{inc}(\tilde{x}, \tilde{y})$. By Proposition 2.1 (ii), we may assume that $p = +\chi_i$ for some $i \in E$. We show that

$$\tilde{x} + \chi_i \in \mathcal{J} \text{ or } \exists q \in \mathrm{inc}(\tilde{x} + \chi_i, \tilde{y}) \text{ such that } \tilde{x} + \chi_i + q \in \mathcal{J}. \tag{3.1}$$

Define $x, y \in \mathcal{F}$ by $x = \pi(\tilde{x})$ and $y = \pi(\tilde{y})$. Then, we have $+\chi_i \in \mathrm{inc}(x, y)$ since $+\chi_i \in \mathrm{inc}(\tilde{x}, \tilde{y})$ and each π_e is a strictly increasing function. Below we consider the following two cases and give a proof of (3.1) for each case:

(Case 1) $\{\alpha \in \mathbb{Z} \mid 0 < \alpha \leq y(i) - x(i),\ x + \alpha\chi_i \in \mathcal{F}\} \neq \emptyset$,
(Case 2) $\{\alpha \in \mathbb{Z} \mid 0 < \alpha \leq y(i) - x(i),\ x + \alpha\chi_i \in \mathcal{F}\} = \emptyset$.

We firstly consider Case 1.

Lemma 3.6. *It holds that*

$$\text{(a) } x + \alpha_1\chi_i \in \mathcal{F}, \quad \textit{or} \quad \text{(b) } x + \alpha_2\chi_i \in \mathcal{F} \textit{ and } y(i) - x(i) \geq \alpha_2 \quad \textit{(or both)},$$

where $\alpha_1 = \pi_i(\tilde{x}(i) + 1) - \pi_i(\tilde{x}(i))$ and $\alpha_2 = \pi_i(\tilde{x}(i) + 2) - \pi_i(\tilde{x}(i))$.

Proof. Let

$$\alpha_* = \min\{\alpha \mid \alpha \in \mathbb{Z},\ 0 < \alpha \leq y(i) - x(i),\ x + \alpha\chi_i \in \mathcal{F}\}.$$

By the definition of π_i, every $x' \in \mathcal{F}$ satisfies neither $\pi_i(\tilde{x}(i)) = x(i) < x'(i) < \pi_i(\tilde{x}(i) + 1)$ nor $\pi_i(\tilde{x}(i) + 1) < x'(i) < \pi_i(\tilde{x}(i) + 2)$. Hence, if $\alpha_* \leq \alpha_2$, α_* is either α_1 or α_2, i.e., (a) or (b) holds.

In the following, we assume, to the contrary, that $\alpha_* > \alpha_2$, and derive a contradiction. This assumption and the definition of α_* imply that

$$x + \alpha'\chi_i \notin \mathcal{F} \qquad \text{for all } \alpha' \in \mathbb{Z} \text{ with } 0 < \alpha' \leq \alpha_2. \tag{3.2}$$

Claim 1: There exists some $q \in U \setminus \{+\chi_i\}$ such that $x + \alpha_1(\chi_i + q) \in \mathcal{F}$.
[Proof of Claim 1] Let $z \in \mathcal{F}$ be a vector with $z(i) = \pi_i(\tilde{x}(i) + 1) = x(i) + \alpha_1$. Since $+\chi_i \in \mathrm{inc}(x, z)$, the property (NS′) implies that there exist some $q \in \mathrm{inc}(x, z) \cup \{\mathbf{0}\} \setminus \{+\chi_i\}$ and $\gamma \in \mathbb{Z}_{++}$ such that $x + \gamma(\chi_i + q) \in \mathcal{F}$ and $x + \gamma(\chi_i + q)$ is between x and z. It follows that $0 < \gamma \leq z(i) - x(i) = \alpha_1$, implying $\gamma = \alpha_1$ since $\{x' \in \mathcal{F} \mid x(i) < x'(i) < x(i) + \alpha_1\} = \emptyset$. By (3.2), we have $q \neq \mathbf{0}$, implying that $q \in U \setminus \{+\chi_i\}$. [End of Claim 1]

Claim 2: $\alpha_* \leq 2\alpha_1$.
[Proof of Claim 2] Assume, to the contrary, that $\alpha_* > 2\alpha_1$. Put $v = x + \alpha_*\chi_i$ ($\in \mathcal{F}$), and consider the vector $v + (\alpha_* - \alpha_1)(-\chi_i) + \alpha_1 q = x + \alpha_1(\chi_i + q) \in \mathcal{F}$, where q is the vector in Claim 1. By Lemma 3.4, there exists some $\varepsilon \in \mathbb{Z}$ with $0 \leq \varepsilon \leq \alpha_1$ such that

$$v + (\alpha_* - \alpha_1 - \varepsilon)(-\chi_i) = x + (\alpha_1 + \varepsilon)\chi_i \in \mathcal{F}.$$

This, however, is a contradiction to the definition of α_* since $\alpha_1 + \varepsilon \leq 2\alpha_1 < \alpha_*$. [End of Claim 2]

Let $z \in \mathcal{F}$ be a vector with $z(i) = \pi_i(\tilde{x}(i) + 2) = x(i) + \alpha_2$. Since $\alpha_* > \alpha_2$, we have $-\chi_i \in \mathrm{inc}(x + \alpha_*\chi_i, z)$. By (NS′), there exists some $s \in \mathrm{inc}(x + \alpha_*\chi_i, z) \cup \{\mathbf{0}\} \setminus \{-\chi_i\}$ and $\gamma \in \mathbb{Z}_{++}$ such that $x + (\alpha_* - \gamma)\chi_i + \gamma s \in \mathcal{F}$ and $x + (\alpha_* - \gamma)\chi_i + \gamma s$ is between $x + \alpha_*\chi_i$ and z, where the latter condition implies $\alpha_2 \leq \alpha_* - \gamma < \alpha_*$. This inequality and the definition of α_* imply $s \neq \mathbf{0}$. We have

$$(\alpha_* - \gamma) - \gamma \geq \alpha_* - 2(\alpha_* - \alpha_2) = -\alpha_* + 2\alpha_2 > -\alpha_* + 2\alpha_1 \geq 0,$$

where the last inequality is by Claim 2. Hence, $\alpha_* - \gamma > \gamma$ holds, and Lemma 3.4 applied to x and $x + (\alpha_* - \gamma)\chi_i + \gamma s$ implies that there exists some $\varepsilon \in \mathbb{Z}$ with $0 \leq \varepsilon \leq \gamma$ such that $x + (\alpha_* - \gamma - \varepsilon)\chi_i \in \mathcal{F}$. This, however, is a contradiction to the definition of α_* since $\alpha_* - \gamma - \varepsilon \leq \alpha_* - 2\gamma < \alpha_*$. □

Lemma 3.6 implies that we have either (a) $\tilde{x} + \chi_i \in \mathcal{J}$ or (b) $\tilde{x} + 2\chi_i \in \mathcal{J}$ and $\tilde{y}(i) - \tilde{x}(i) \geq 2$ (or both), i.e., (3.1) holds in Case 1.

We next consider Case 2. That is, we assume

$$x + \alpha\chi_i \notin \mathcal{F} \qquad (0 < \forall\alpha \leq y(i) - x(i)), \tag{3.3}$$

and show that (3.1) holds. The assumption (3.3) implies, in particular, $x + \alpha_1\chi_i \notin \mathcal{F}$ since $\alpha_1 \leq y(i) - x(i)$.

Since $+\chi_i \in \mathrm{inc}(x, y)$, the property (NS′) implies that there exist $q \in \mathrm{inc}(x, y) \cup \{\mathbf{0}\} \setminus \{+\chi_i\}$ and $\beta \in \mathbb{Z}_{++}$ such that $x + \beta(\chi_i + q)$ is a neighbor of x and between x and y. Note that $\beta \leq y(i) - x(i)$, which, together with (3.3), implies $q \neq \mathbf{0}$. By Proposition 2.1 (ii), we may assume that $q = +\chi_j$ for some $j \in E \setminus \{i\}$. Since $x + \beta(\chi_i + \chi_j)$ is a neighbor of x, we have

$$x + \alpha(\chi_i + \chi_j) \notin \mathcal{F} \qquad (0 < \forall\alpha < \beta). \tag{3.4}$$

Lemma 3.7. *Let $\beta', \beta'' \in \mathbb{Z}$ be integers such that $0 \leq \beta' \leq \beta$, $0 \leq \beta'' \leq \beta$, and $x + \beta'\chi_i + \beta''\chi_j \in \mathcal{F}$. Then, we have $(\beta', \beta'') \in \{(0,0), (0,\beta), (\beta,\beta)\}$.*

Proof. It suffices to show that $\beta', \beta'' \in \{0, \beta\}$ since $x + \beta\chi_i \notin \mathcal{F}$ by (3.3). Assume, to the contrary, that $0 < \beta' < \beta$ holds. Since $+\chi_i \in \mathrm{inc}(x, x + \beta'\chi_i + \beta''\chi_j)$, the property (NS′) implies that there exists some $\eta \in \mathbb{Z}$ with $0 < \eta \leq \beta' < \beta$ such that either $x + \eta(\chi_i + \chi_j) \in \mathcal{F}$ or $x + \eta\chi_i \in \mathcal{F}$ (or both). This, however, contradicts (3.3) or (3.4).

We then assume, to the contrary, that $0 < \beta'' < \beta$ holds. We have $-\chi_j \in \mathrm{inc}(x + \beta(\chi_i + \chi_j), x + \beta'\chi_i + \beta''\chi_j)$. Therefore, the property (NS′) implies that there exists some $\eta \in \mathbb{Z}$ with $0 < \eta \leq \beta - \beta'' < \beta$ such that either $x + (\beta - \eta)(\chi_i + \chi_j) \in \mathcal{F}$ or $x + \beta\chi_i + (\beta - \eta)\chi_j \in \mathcal{F}$ (or both). By (3.4), we have $x + \beta\chi_i + (\beta - \eta)\chi_j \in \mathcal{F}$. Lemma 3.4 applied to x and $x + \beta\chi_i + (\beta - \eta)\chi_j$ implies that there exists $\varepsilon \in \mathbb{Z}$ with $0 \leq \varepsilon \leq \beta - \eta$ such that and $x + (\beta - \varepsilon)\chi_i \in \mathcal{F}$, a contradiction to (3.3). □

We then prove

$$\beta = \pi_i(\tilde{x}(i) + 1) - \pi_i(\tilde{x}(i)) = \pi_j(\tilde{x}(j) + 1) - \pi_j(\tilde{x}(j)), \tag{3.5}$$

which follows from the definitions of π_i, π_j and the next lemma.

Lemma 3.8. (i) $\{z \in \mathcal{F} \mid 0 < z(i) - x(i) < \beta\} = \emptyset$, (ii) $\{z \in \mathcal{F} \mid 0 < z(j) - x(j) < \beta\} = \emptyset$.

Proof. [Proof of (i)] Assume, to the contrary, that there exists some $z \in \mathcal{F}$ such that $x(i) < z(i) < x(i) + \beta$. Since $\chi_i \in \mathrm{inc}(x, z)$, the property (NS$'$) implies that there exist some $s \in \mathrm{inc}(x, z) \cup \{\mathbf{0}\} \setminus \{+\chi_i\}$ and $\gamma \in \mathbb{Z}$ such that

$$0 < \gamma \leq z(i) - x(i) < \beta \text{ and } x' \equiv x + \gamma(\chi_i + s) \in \mathcal{F}.$$

By the assumption (3.3), we have $s \neq \mathbf{0}$.

Suppose that $s \in \{+\chi_j, -\chi_j\}$. Since $-\chi_i \in \mathrm{inc}(x + \beta(\chi_i + \chi_j), x')$, the property (NS$'$) implies that there exists some $\eta \in \mathbb{Z}$ with $0 < \eta \leq \beta - \gamma < \beta$ such that

$$\text{either } x + (\beta - \eta)\chi_i + \beta\chi_j \in \mathcal{F} \text{ or } x + (\beta - \eta)(\chi_i + \chi_j) \in \mathcal{F} \text{ (or both).}$$

This, however, is a contradiction to Lemma 3.7. Hence, we have $s \notin \{+\chi_j, -\chi_j\}$, implying that $s \in U \setminus \{\pm\chi_i, \pm\chi_j\}$.

Lemma 3.5 applied to $x+\beta(\chi_i+\chi_j)$ and $x+\gamma(\chi_i+s)$ implies that there exist some $\varepsilon_1, \varepsilon_2 \in \mathbb{Z}$ such that

$$0 \leq \varepsilon_1 \leq \beta - \gamma,\ 0 \leq \varepsilon_2 \leq \beta,\ \varepsilon_1 + \varepsilon_2 \leq \gamma, \text{ and } x + (\gamma + \varepsilon_1)\chi_i + \varepsilon_2\chi_j \in \mathcal{F}.$$

This, however, is a contradiction to Lemma 3.7 since $0 < \gamma \leq \gamma + \varepsilon_1 \leq \beta$. This concludes the proof of (i).

[Proof of (ii)] Assume, to the contrary, that there exists some $z \in \mathcal{F}$ such that $x(j) < z(j) < x(j) + \beta$. Since $+\chi_j \in \mathrm{inc}(x, z)$, the property (NS$'$) implies that there exist some $s \in \mathrm{inc}(x, z) \cup \{\mathbf{0}\} \setminus \{+\chi_j\}$ and $\gamma \in \mathbb{Z}$ such that

$$0 < \gamma \leq z(j) - x(j) < \beta \text{ and } x' \equiv x + \gamma(\chi_j + s) \in \mathcal{F}.$$

In a similar way as in the proof of (i), we can show that $s \in U \setminus \{\pm\chi_i, \pm\chi_j\}$.

Since $-\chi_j \in \mathrm{inc}(x + \beta(\chi_i + \chi_j), x')$, the property (NS$'$) implies that there exists some $\eta \in \mathbb{Z}_{++}$ such that at least one of (a), (b), or (c) holds, where

(a) $x + \beta\chi_i + (\beta - \eta)\chi_j \in \mathcal{F}$ and $\eta \leq \beta - \gamma$,
(b) $x + (\beta - \eta)(\chi_i + \chi_j) \in \mathcal{F}$ and $\eta \leq \beta - \gamma$,
(c) $x + \beta\chi_i + (\beta - \eta)\chi_j + \eta s \in \mathcal{F}$ and $\eta \leq \min\{\beta - \gamma, \gamma\}$.

By Lemma 3.7, we cannot have (a) and (b), and therefore (c) holds. By Lemma 3.5 applied to x and $x+\beta\chi_i+(\beta-\eta)\chi_j+\eta s$, there exist some $\varepsilon_1, \varepsilon_2 \in \mathbb{Z}$ such that $0 \leq \varepsilon_1 \leq \beta$, $0 \leq \varepsilon_2 \leq \beta-\eta$, $\varepsilon_1 + \varepsilon_2 \leq \eta$, and

$$x + (\beta - \varepsilon_1)\chi_i + (\beta - \eta - \varepsilon_2)\chi_j \in \mathcal{F}.$$

This, however, is a contradiction to Lemma 3.7 since $\beta - \eta - \varepsilon_2 < \beta$ and $\beta - \varepsilon_1 \geq \beta - \eta \geq \gamma > 0$. $\square$

Since $x + \beta(\chi_i + \chi_j) \in \mathcal{F}$, the equation (3.5) implies that $\tilde{x} + \chi_i + \chi_j \in \mathcal{J}$. We have $+\chi_j \in \mathrm{inc}(\tilde{x}, \tilde{y}) \setminus \{+\chi_i\} \subseteq \mathrm{inc}(\tilde{x} + \chi_i, \tilde{y})$, and therefore (3.1) holds. This concludes the proof of Lemma 3.2 (i).

3.2.2 Proof of Lemma 3.2 (ii)

Let $x, y \in \mathcal{F}$, and put $\tilde{x} = \pi^{-1}(x), \tilde{y} = \pi^{-1}(y)$. By definition, it holds that $\tilde{x}, \tilde{y} \in \mathcal{J}$.

Proof of "if" part We show that if $\tilde{y}$ is a proper neighbor of $\tilde{x}$ in $\mathcal{J}$, then y is a proper neighbor of x in $\mathcal{F}$.

[Case 1: $|\mathrm{supp}(\tilde{x} - \tilde{y})| = 1$] Let $i \in E$ be the unique element in $\{e \in E \mid \tilde{x}(e) \neq \tilde{y}(e)\}$. We may assume that $\tilde{x}(i) < \tilde{y}(i)$. Then, we have either

$$\text{(a) } \tilde{y} = \tilde{x} + \chi_i \in \mathcal{J}, \qquad \text{or} \qquad \text{(b) } \tilde{y} = \tilde{x} + 2\chi_i \in \mathcal{J} \text{ and } \tilde{x} + \chi_i \notin \mathcal{J}.$$

If (a) holds, then we have $y = x + \alpha_1\chi_i \in \mathcal{F}$ with $\alpha_1 = \pi_i(\tilde{x}(i)+1) - \pi_i(\tilde{x}(i))$, which is a proper neighbor of x in $\mathcal{F}$. If (b) holds, then we have $y = x+\alpha_2\chi_i \in \mathcal{F}$ with $\alpha_2 = \pi_i(\tilde{x}(i)+2) - \pi_i(\tilde{x}(i))$ and $x + \alpha_1\chi_i \notin \mathcal{F}$, implying that y is a proper neighbor of x in $\mathcal{F}$.

[Case 2: $|\mathrm{supp}(\tilde{x} - \tilde{y})| = 2$] We may assume, without loss of generality, that $\tilde{y} = \tilde{x}+\chi_i+\chi_j$ for some distinct $i, j \in E$. Since $\tilde{y}$ is a proper neighbor of $\tilde{x}$ in $\mathcal{J}$, we may also assume that $\tilde{x} + \chi_i \notin \mathcal{J}$. Then, we have $y = x + \alpha\chi_i + \beta\chi_j \in \mathcal{F}$ and $x + \alpha\chi_i \notin \mathcal{F}$, where $\alpha = \pi_i(\tilde{x}(i)+1) - \pi_i(\tilde{x}(i))$, and $\beta = \pi_j(\tilde{x}(j)+1) - \pi_j(\tilde{x}(j))$. By the assumption and the definitions of α, β, the following property holds:

$$\text{if } x + \alpha'\chi_i + \beta'\chi_j \in \mathcal{F},\ 0 \leq \alpha' \leq \alpha,\ 0 \leq \beta' \leq \beta, \text{ then } (\alpha', \beta') \in \{(0,0), (0,\beta), (\alpha,\beta)\}. \tag{3.6}$$

Hence, y is a proper neighbor of x in $\mathcal{F}$ if $\alpha = \beta$.

By the property (NS) applied to x, $x + \alpha\chi_i + \beta\chi_j$, and $i \in \mathrm{inc}(x, x + \alpha\chi_i + \beta\chi_j)$, there exists some $\gamma \in \mathbb{Z}$ with $0 < \gamma \leq \min\{\alpha, \beta\}$ such that $x + \gamma\chi_i \in \mathcal{F}$ or $x + \gamma(\chi_i + \chi_j) \in \mathcal{F}$. Since $\gamma \leq \alpha$, we cannot have $x + \gamma\chi_i \in \mathcal{F}$. Hence, $x + \gamma(\chi_i + \chi_j) \in \mathcal{F}$ holds. By the definitions of α, β, we have $\gamma \geq \max\{\alpha, \beta\}$, implying that $\gamma = \alpha = \beta$. This concludes that y is a proper neighbor of x in $\mathcal{F}$.

Proof of "only if" part We show that if y is a proper neighbor of x in $\mathcal{F}$, then $\tilde{y}$ is a proper neighbor of $\tilde{x}$ in $\mathcal{J}$.

[Case 1: $|\mathrm{supp}(x - y)| = 1$] Let $i \in E$ be the unique element in $\{e \in E \mid x(e) \neq y(e)\}$. We may assume that $x(i) < y(i)$. Then, there exists some $\alpha_* \in \mathbb{Z}_{++}$ such that

$$y = x + \alpha_*\chi_i \in \mathcal{F}, \qquad x + \alpha'\chi_i \notin \mathcal{F} \quad (0 < \forall\alpha' < \alpha_*).$$

If $\alpha_* = \pi_i(\tilde{x}(i) + 1) - \pi_i(\tilde{x}(i))$, then $\tilde{y} = \pi^{-1}(x + \alpha_*\chi_i) = \tilde{x} + \chi_i \in \mathcal{J}$, which is a proper neighbor of $\tilde{x}$ in $\mathcal{J}$ since $+\chi_i \in \mathrm{inc}(\tilde{x}, \tilde{y})$. Hence, suppose that $\alpha_* > \pi_i(\tilde{x}(i) + 1) - \pi_i(\tilde{x}(i))$. Then, Lemma 3.6 implies that $\alpha_* = \pi_i(\tilde{x}(i) + 2) - \pi_i(\tilde{x}(i))$. Therefore, it holds that

$$\tilde{y} = \pi^{-1}(x + \alpha_*\chi_i) = \tilde{x} + 2\chi_i \in \mathcal{J}, \qquad \tilde{x} + \chi_i \notin \mathcal{J},$$

which shows that $\tilde{y}$ is a proper neighbor of $\tilde{x}$ in $\mathcal{J}$.

[Case 2: $|\mathrm{supp}(x - y)| = 2$] We may assume, without loss of generality, that $y = x + \alpha(\chi_i + \chi_j)$ for some distinct $i, j \in E$ and $\alpha \in \mathbb{Z}_{++}$. Since y is a proper neighbor of x in $\mathcal{F}$, we may also assume that

$$x + \alpha'\chi_i \notin \mathcal{F} \qquad (0 < \forall\alpha' < \alpha).$$

Then, in the same way as in the proof of Lemma 3.8, we can show that

$$\alpha = \pi_i(\tilde{x}(i) + 1) - \pi_i(\tilde{x}(i)) = \pi_j(\tilde{x}(j) + 1) - \pi_j(\tilde{x}(j))$$

(cf. (3.5)). This implies $\tilde{y} = \pi^{-1}(x + \alpha(\chi_i + \chi_j)) = \tilde{x} + \chi_i + \chi_j \in \mathcal{J}$ and $\tilde{x} + \chi_i \notin \mathcal{J}$. Therefore, $\tilde{y}$ is a proper neighbor of $\tilde{x}$ in $\mathcal{J}$.

4 Polyhedral Structure of Neighbor Systems

We consider the convex closure of neighbor systems, and reveal their polyhedral structure by showing the relationship with bisubmodular polyhedra.

4.1 Results

For a set $\mathcal{F} \subseteq \mathbb{Z}^E$, the *convex closure (or closed convex hull)* of $\mathcal{F}$, denoted by $\mathrm{conv}(\mathcal{F})$ ($\subseteq \mathbb{R}^E$) is the (unique) minimal closed convex set containing $\mathcal{F}$. If $\mathcal{F}$ is a finite set, then $\mathrm{conv}(\mathcal{F})$ coincides with the *convex hull* of $\mathcal{F}$, which is the set of vectors represented as a convex combination of finite vectors in $\mathcal{F}$.

We denote

$$2^E = \{X \mid X \subseteq E\}, \quad 3^E = \{(X, Y) \mid X, Y \subseteq E,\ X \cap Y = \emptyset\}.$$

For $x \in \mathbb{R}^E$ and $X \in 2^E$, we define $x(X) = \sum_{i \in X} x(i)$. Similarly, for $x \in \mathbb{R}^E$ and $(X, Y) \in 3^E$, we define $x(X, Y) = x(X) - x(Y)$. A function $\rho : 3^E \to \mathbb{R} \cup \{+\infty\}$ is said to be *bisubmodular* if it satisfies the bisubmodular inequality:

$$\begin{aligned}
&\rho(X_1, Y_1) + \rho(X_2, Y_2) \\
&\quad \geq \ \rho((X_1 \cup X_2) \setminus (Y_1 \cup Y_2), (Y_1 \cup Y_2) \setminus (X_1 \cup X_2)) + \rho(X_1 \cap X_2, Y_1 \cap Y_2) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\forall (X_1, Y_1), (X_2, Y_2) \in 3^E).
\end{aligned}$$

For a function $\rho : 3^E \to \mathbb{R} \cup \{+\infty\}$ with $\rho(\emptyset, \emptyset) = 0$, we define a polyhedron $\mathrm{P}_*(\rho) \subseteq \mathbb{R}^E$ by

$$\mathrm{P}_*(\rho) = \{x \in \mathbb{R}^E \mid x(X, Y) \leq \rho(X, Y) \ ((X, Y) \in 3^E)\}.$$

The polyhedron $\mathrm{P}_*(\rho)$ is called a *bisubmodular polyhedron* if ρ is a bisubmodular function. A polyhedron $P \subseteq \mathbb{R}^E$ is said to be *integral* if it satisfies $\mathrm{conv}(P \cap \mathbb{Z}^E) = P$. For a bounded polyhedron P, we have $\mathrm{conv}(P \cap \mathbb{Z}^E) = P$ if and only if all extreme points in P are integral vectors.

The following result is known for the convex closure of a jump system, which is a special case of neighbor systems.

Theorem 4.1 ([6, Theorem 5.3])**.** *For every jump system $\mathcal{J} \subseteq \mathbb{Z}^E$, its convex closure* $\mathrm{conv}(\mathcal{J})$ *is an integral bisubmodular polyhedron.*

We show that this result extends to neighbor systems.

Theorem 4.2. *For every all-neighbor system $\mathcal{F} \subseteq \mathbb{Z}^E$, its convex closure* $\mathrm{conv}(\mathcal{F})$ *is an integral bisubmodular polyhedron.*

It should be noted that Theorem 4.2 does not follow from Theorems 3.1 and 4.1.

We also provide a characterization of neighbor systems by the property that the convex closure is a bisubmodular polyhedron.

Theorem 4.3. *A nonempty set $\mathcal{F} \subseteq \mathbb{Z}^E$ is an all-neighbor system if and only if for all vectors $\ell, u \in \mathbb{Z}^E$ satisfying $\ell \leq u$ and $\mathcal{F} \cap [\ell, u] \neq \emptyset$, the convex closure* $\mathrm{conv}(\mathcal{F} \cap [\ell, u])$ *is an integral bisubmodular polyhedron.*

In the next section we give proofs of Theorems 4.2 and 4.3.

4.2 Proofs

4.2.1 Proof of Theorem 4.2

Let $\mathcal{F} \subseteq \mathbb{Z}^E$ be a neighbor system, and $\rho : 3^E \to \mathbb{R} \cup \{+\infty\}$ is a function defined by

$$\rho(X, Y) = \sup\{x(X, Y) \mid x \in \mathcal{F}\} \qquad ((X, Y) \in 3^E).$$

By the definition of ρ, the following properties hold:

$$\begin{aligned} &x(X, Y) \leq \rho(X, Y) \quad (\forall x \in \mathcal{F},\ \forall (X, Y) \in 3^E), \\ &\rho(\emptyset, \emptyset) = 0, \quad \rho(X, Y) \in \mathbb{Z} \text{ if } \rho(X, Y) < +\infty. \end{aligned} \tag{4.1}$$

We use the following property of bisubmodular polyhedra to prove that $\mathrm{conv}(\mathcal{F})$ is an integral bisubmodular polyhedron.

Theorem 4.4 ([6, Corollary 5.4]; see also [16, 20, 21])**.** *If ρ is an integer-valued bisubmodular function, then the bisubmodular polyhedron $\mathrm{P}_*(\rho)$ is an integral polyhedron.*

By Theorem 4.4, it suffices to show the following property holds:

$$\rho \text{ is a bisubmodular function satisfying } \mathrm{conv}(\mathcal{F}) = \mathrm{P}_*(\rho). \tag{4.2}$$

We firstly consider the case where $\mathcal{F}$ is a finite set. In this case, $\rho(X, Y) < +\infty$ holds for every $(X, Y) \in 3^E$. We give a key property to show (4.2).

Lemma 4.5. *For every $(A, B) \in 3^E$ with $A \cup B = E$ and all k-tuples of subsets $E_1, E_2, \ldots, E_k$ ($k \geq 1$) of E with $E_1 \subset E_2 \subset \cdots \subset E_k$, there exists some $x \in \mathcal{F}$ such that*

$$x(E_t \cap A, E_t \cap B) = \rho(E_t \cap A, E_t \cap B) \qquad (t = 1, 2, \ldots, k).$$

Proof. We prove the claim by induction on k. The case where $k = 1$ is obvious from the definition of ρ. Hence, we assume $k > 1$. By the induction hypothesis, there exists some $x \in \mathcal{F}$ such that

$$x(E_t \cap A, E_t \cap B) = \rho(E_t \cap A, E_t \cap B) \qquad (t = 1, 2, \ldots, k - 1).$$

Let $y \in \mathcal{F}$ be a vector satisfying $y(E_k \cap A, E_k \cap B) = \rho(E_k \cap A, E_k \cap B)$, and assume that y minimizes the value $||y - x||_1$ among all such y. We will show that y satisfies

$$y(E_t \cap A, E_t \cap B) = \rho(E_t \cap A, E_t \cap B) \qquad (t = 1, 2, \ldots, k - 1).$$

Assume, to the contrary, that there exists some $t \in \{1, 2, \ldots, k - 1\}$ such that $y(E_t \cap A, E_t \cap B) < \rho(E_t \cap A, E_t \cap B)$. Since $x(E_t \cap A, E_t \cap B) = \rho(E_t \cap A, E_t \cap B)$, we have either $\{e \in E \mid e \in E_t \cap A,\ x(e) > y(e)\} \neq \emptyset$ or $\{e \in E \mid e \in E_t \cap B,\ x(e) < y(e)\} \neq \emptyset$ (or both). We consider the former case only since the latter case can be dealt with in a similar way.

Let $i \in E$ be an element such that $i \in E_t \cap A$ and $x(i) > y(i)$. Since $+\chi_i \in \mathrm{inc}(y, x)$, the property (NS′) implies that there exist $q \in \mathrm{inc}(y, x) \cup \{\mathbf{0}\} \setminus \{+\chi_i\}$ and $\alpha \in \mathbb{Z}_{++}$ such that $y' = y + \alpha(\chi_i + q) \in \mathcal{F}$ and y' is between y and x. We note that $||y' - x||_1 < ||y - x||_1$ holds. If $q = \mathbf{0}$, then we have

$$y'(E_k \cap A, E_k \cap B) > y(E_k \cap A, E_k \cap B) = \rho(E_k \cap A, E_k \cap B)$$

since $i \in E_t \cap A \subseteq E_k \cap A$, a contradiction. Hence, $q \neq \mathbf{0}$ holds. From this follows that

$$y'(E_k \cap A, E_k \cap B) \geq y(E_k \cap A, E_k \cap B) = \rho(E_k \cap A, E_k \cap B),$$

where the inequality must hold with equality by (4.1). In addition, we have $||y'-x||_1 < ||y-x||_1$, a contradiction to the choice of y. □

To show the bisubmodularity of ρ, we use the following characterization.

Lemma 4.6 ([4, Theorem 2]). *A function $\rho : 3^E \to \mathbb{R}$ is bisubmodular if and only if ρ satisfies the following conditions:*

$$\begin{aligned}
&\rho(X \cap A, X \cap B) + \rho(Y \cap A, Y \cap B) \\
&\quad \geq \ \rho((X \cup Y) \cap A, (X \cup Y) \cap B) + \rho((X \cap Y) \cap A, (X \cap Y) \cap B) \\
&\qquad\qquad (\forall (A, B) \in 3^E \text{ with } A \cup B = E, \forall X, Y \in 2^E), \qquad (4.3) \\
&\rho(X \cup \{e\}, Y) + \rho(X, Y \cup \{e\}) \ \geq \ 2\rho(X, Y) \\
&\qquad\qquad (\forall (X, Y) \in 3^E,\ \forall e \in E \setminus (X \cup Y)). \qquad (4.4)
\end{aligned}$$

Note that the condition (4.3) is equivalent to the submodularity of the function $\rho_{A,B} : 2^E \to \mathbb{R}$ defined by $\rho_{A,B}(X) = \rho(X \cap A, X \cap B)$ $(X \in 2^E)$. By using this characterization, we prove that the function ρ is bisubmodular.

Lemma 4.7. *The function ρ is bisubmodular.*

Proof. By Lemma 4.6 it suffices to show that ρ satisfies the conditions (4.3) and (4.4).

We firstly show the condition (4.3). By Lemma 4.5, there exists a vector $x \in \mathcal{F}$ satisfying

$$\begin{aligned}
x((X \cap Y) \cap A, (X \cap Y) \cap B) &= \rho((X \cap Y) \cap A, (X \cap Y) \cap B), \\
x(X \cap A, X \cap B) &= \rho(X \cap A, X \cap B), \\
x((X \cup Y) \cap A, (X \cup Y) \cap B) &= \rho((X \cup Y) \cap A, (X \cup Y) \cap B),
\end{aligned}$$

which, together with (4.1), implies the desired inequality as follows:

$$\begin{aligned}
&\rho((X \cup Y) \cap A, (X \cup Y) \cap B) + \rho((X \cap Y) \cap A, (X \cap Y) \cap B) \\
&= \ x((X \cup Y) \cap A, (X \cup Y) \cap B) + x((X \cap Y) \cap A, (X \cap Y) \cap B) \\
&= \ x(X \cap A, X \cap B) + x(Y \cap A, Y \cap B) \\
&\leq \ \rho(X \cap A, X \cap B) + \rho(Y \cap A, Y \cap B).
\end{aligned}$$

We then show the condition (4.4). By the definition of ρ, there exists a vector $x \in \mathcal{F}$ satisfying $x(X, Y) = \rho(X, Y)$, which, together with (4.1), implies the desired inequality:

$$\begin{aligned}
\rho(X \cup \{e\}, Y) + \rho(X, Y \cup \{e\}) &\geq \{x(X) + x(e) - x(Y)\} + \{x(X) - x(Y) - x(e)\} \\
&= 2\{x(X) - x(Y)\} = \rho(X, Y).
\end{aligned}$$

□

To show the equation $\mathrm{conv}(\mathcal{F}) = \mathrm{P}_*(\rho)$, we use the following characterization of extreme points in a bounded bisubmodular polyhedron.

Lemma 4.8 ([12, Corollary 3.59])**.** *Let $\rho : 3^E \to \mathbb{R}$ be a bisubmodular function. A vector $x \in \mathbb{R}^E$ is an extreme point of* $\mathrm{P}_*(\rho)$ *if and only if there exist $(A, B) \in 3^E$ with $A \cup B = E$ and an ordering $e_1, e_2, \ldots, e_n$ of elements in E such that*

$$x(E_t \cap A, E_t \cap B) = \rho(E_t \cap A, E_t \cap B) \qquad (t = 1, 2, \ldots, n),$$

where $E_t = \{e_1, e_2, \ldots, e_t\}$.

Lemma 4.9. *It holds that* $\mathrm{conv}(\mathcal{F}) = \mathrm{P}_*(\rho)$.

Proof. By the definition of $\mathrm{P}_*(\rho)$, it is easy to see that $\mathrm{conv}(\mathcal{F}) \subseteq \mathrm{P}_*(\rho)$. To show the reverse inclusion, it suffices to show that every extreme point of $\mathrm{P}_*(\rho)$ is contained in $\mathcal{F}$, which follows from Lemmas 4.5 and 4.8. $\square$

This concludes the proof of (4.2) for the case where $\mathcal{F}$ is a finite neighbor system.

We then prove (4.2) for the case where a neighbor system $\mathcal{F}$ is an infinite set. With a fixed vector $x_0 \in \mathcal{F}$, we define $\mathcal{F}_k$ $(k = 1, 2, \ldots)$ by

$$\mathcal{F}_k = \{x \in \mathcal{F} \mid |x(e) - x_0(e)| \leq k \ (\forall e \in E)\}.$$

Note that $\mathcal{F}_k$ is a finite set. We also define a function $\rho_k : 3^E \to \mathbb{Z}$ $(k = 1, 2, \ldots)$ by

$$\rho_k(X, Y) = \max\{x(X, Y) \mid x \in \mathcal{F}_k\} \qquad ((X, Y) \in 3^E).$$

By Proposition 2.1 (iii), each $\mathcal{F}_k$ is a neighbor system, and therefore ρ_k is a bisubmodular function by Lemma 4.7. Moreover, it holds that $\lim_{k\to+\infty} f_k(X, Y) = f(X, Y)$ for every $(X, Y) \in 3^E$. Therefore, for every $(X_1, Y_1), (X_2, Y_2) \in 3^E$, we have

$$\begin{aligned}
&\rho(X_1, Y_1) + \rho(X_2, Y_2) \\
&\quad = \lim_{k\to+\infty} \rho_k(X_1, Y_1) + \lim_{k\to+\infty} \rho_k(X_2, Y_2) \\
&\quad \geq \lim_{k\to+\infty} \rho_k((X_1 \cup X_2) \setminus (Y_1 \cup Y_2), (Y_1 \cup Y_2) \setminus (X_1 \cup X_2)) + \lim_{k\to+\infty} \rho_k(X_1 \cap X_2, Y_1 \cap Y_2) \\
&\quad = \rho((X_1 \cup X_2) \setminus (Y_1 \cup Y_2), (Y_1 \cup Y_2) \setminus (X_1 \cup X_2)) + \rho(X_1 \cap X_2, Y_1 \cap Y_2),
\end{aligned}$$

i.e., ρ is a bisubmodular function.

We then show that $\mathrm{conv}(\mathcal{F}) = \mathrm{P}_*(\rho)$ holds. Since $\mathcal{F} \subseteq \mathrm{P}_*(\rho)$, we have $\mathrm{conv}(\mathcal{F}) \subseteq \mathrm{P}_*(\rho)$. It holds that $\mathrm{P}_*(\rho) = \lim_{k\to+\infty} \mathrm{P}_*(\rho_k)$, and that $\mathrm{P}_*(\rho_k) = \mathrm{conv}(\mathcal{F}_k)$ for each $k = 1, 2, \ldots$ by Lemma 4.9. Since $\mathrm{conv}(\mathcal{F}_k) \subseteq \mathrm{conv}(\mathcal{F})$, we have

$$\mathrm{P}_*(\rho) = \lim_{k\to+\infty} \mathrm{P}_*(\rho_k) = \lim_{k\to+\infty} \mathrm{conv}(\mathcal{F}_k) \subseteq \mathrm{conv}(\mathcal{F}).$$

Hence, $\mathrm{conv}(\mathcal{F}) = \mathrm{P}_*(\rho)$ holds. This concludes the proof of (4.2) for the case where $\mathcal{F}$ is an infinite neighbor system.

4.2.2 Proof of Theorem 4.3

We show that a nonempty set $\mathcal{F} \subseteq \mathbb{Z}^E$ is an all-neighbor system if and only if the convex closure $\mathrm{conv}(\mathcal{F} \cap [\ell, u])$ is an integral bisubmodular polyhedron for all vectors $\ell, u \in \mathbb{Z}^E$ satisfying $\ell \leq u$ and $\mathcal{F} \cap [\ell, u] \neq \emptyset$.

[Proof of "only if" part] Let $\mathcal{F}$ be an all-neighbor system. By Proposition 2.1 (iii), $\mathcal{F} \cap [\ell, u]$ is also an all-neighbor system for all vectors $\ell, u \in \mathbb{Z}^E$ satisfying $\ell \leq u$ and $\mathcal{F} \cap [\ell, u] \neq \emptyset$. Therefore, the "only if" part follows immediately from Theorem 4.2.

[Proof of "if" part] Let $x, y \in \mathcal{F}$ and $i \in \mathrm{supp}(x - y)$. Assume, without loss of generality, that $x(i) < y(i)$. We show that there exists a neighbor $z \in \mathcal{F}$ of x such that z is between x and y and $z(i) > x(i)$.

We define vectors $\ell, u \in \mathbb{Z}^E$ by

$$\ell(e) = \min\{x(e), y(e)\}, \quad u(e) = \max\{x(e), y(e)\} \qquad (e \in E).$$

The set $\mathcal{F} \cap [\ell, u]$ is nonempty since $x \in \mathcal{F} \cap [\ell, u]$. Let $S = \mathrm{conv}(\mathcal{F} \cap [\ell, u])$, which is an integral bisubmodular polyhedron by assumption. By the definitions of ℓ and u, the vector x is an extreme point of S. We consider the *tangent cone* $\mathrm{TC}(x)$ of S at x, which is given by

$$\mathrm{TC}(x) = \{\alpha z \mid z \in \mathbb{R}^E,\ x + z \in S,\ \alpha \in \mathbb{R},\ \alpha \geq 0\}.$$

Since S is a polyhedron, $\mathrm{TC}(x)$ is a polyhedral cone.

Theorem 4.10 ([1, Theorem 3.5])**.** *Let $S \subseteq \mathbb{R}^E$ be a bisubmodular polyhedron, and $x \in S$. Then, every extreme ray of the tangent cone $\mathrm{TC}(x)$ is a positive multiple of a $\{0, +1, -1\}$-vector with exactly one or two nonzero components.*

Since $y - x \in \mathrm{TC}(x)$ and $y(i) > x(i)$, Theorem 4.10 implies that there exists an extreme ray $d \in \mathbb{R}^E$ of $\mathrm{TC}(x)$ that is a positive multiple of a vector $+\chi_i$, $+\chi_i + \chi_k$, or $+\chi_i - \chi_k$ for some $k \in E \setminus \{i\}$. Since S is a bounded polyhedron, there exists some vector $z_0 \in S$ such that z_0 is an extreme point of S and $z_0 - x$ is a positive multiple of d. Since $d(i) > 0$, we have $z_0(i) > x(i)$. The vector z_0 is contained in $\mathcal{F} \cap [\ell, u]$ since it is an extreme point of $S = \mathrm{conv}(\mathcal{F} \cap [\ell, u])$. This implies, in particular, $z_0 \in \mathcal{F}$ and z_0 is between x and y. Let α be the minimum positive number such that $\alpha z_0 + (1 - \alpha)x \in \mathcal{F}$, and put $z = \alpha z_0 + (1 - \alpha)x$. Then, z is a neighbor of x between x and y and satisfies $z(i) > x(i)$. This concludes the proof for the "if" part.

5 Linear Optimization on Neighbor Systems

We consider a linear optimization problem on a finite all-neighbor system $\mathcal{F}$, which is formulated as follows:

(Lin) Maximize $\sum_{e \in E} w(e)x(e)$ subject to $x \in \mathcal{F}$,

where $w \in \mathbb{R}^E$. By using the result in Section 4, we show in Section 5.1 that a greedy algorithm below works for the linear optimization.

Greedy Algorithm for Linear Optimization

Step 0: Let x_0 be any vector in $\mathcal{F}$ and put $x := x_0$. Order the elements in $E = \{e_1, e_2, \ldots, e_n\}$ so that

$$|w(e_1)| \geq |w(e_2)| \geq \cdots \geq |w(e_n)|.$$

Step 1: For $i = 1, 2, \ldots, n$, do the following: if $w(e_i) \geq 0$ (resp., $w(e_i) < 0$), then fix the components $x(e_1), x(e_2), \ldots, x(e_{i-1})$ and decrease (resp., increase) $x(e_i)$ as much as possible

under the condition $x \in \mathcal{F}$.
Step 2: Output the current vector x.

We note that the greedy algorithm is essentially equivalent to a variant of the greedy algorithm (called the "altered greedy algorithm") proposed by Hartvigsen [14], although they seem different. We discuss the relationship between the two greedy algorithms in Section 5.2.

5.1 Validity of Greedy Algorithm

We firstly show the validity of the greedy algorithm above. We define a function $\rho : 3^E \to \mathbb{Z}$ by

$$\rho(X, Y) = \max\{x(X, Y) \mid x \in \mathcal{F}\} \qquad ((X, Y) \in 3^E).$$

Theorem 4.2 and its proof show that ρ is an integer-valued bisubmodular function with $\rho(\emptyset, \emptyset) = 0$ and the convex closure $\mathrm{conv}(\mathcal{F})$ is an integral bisubmodular polyhedron with $\mathrm{conv}(\mathcal{F}) = \mathrm{P}_*(\rho)$. For $i = 1, 2, \ldots, n$, we define $X_i, Y_i \subseteq E$ by

$$X_i = \{e_h \mid 1 \leq h \leq i,\ w(e_h) \geq 0\}, \quad Y_i = \{e_h \mid 1 \leq h \leq i,\ w(e_h) < 0\}.$$

Hence, $\{X_i, Y_i\}$ is a partition of the set $\{e_1, e_2, \ldots, e_i\}$ for each i. It is known that the vector $x_* \in \mathbb{R}^E$ given by

$$x_*(e_i) = \rho(X_i, Y_i) - \rho(X_{i-1}, Y_{i-1}) \qquad (i = 1, 2, \ldots, n)$$

is a vector in the bisubmodular polyhedron $\mathrm{conv}(\mathcal{F})$ maximizing the value $\sum_{e \in E} w(e)x(e)$ among all vectors in $\mathrm{conv}(\mathcal{F})$ (see [9], [12, §3.5 (b)]). Since x_* is an extreme point of $\mathrm{conv}(\mathcal{F})$, x_* is a vector in $\mathcal{F}$; moreover, it is an optimal solution of (Lin). Below we show that x_* is the vector found by the greedy algorithm.

Let $x_a \in \mathcal{F}$ be the vector found by the greedy algorithm, and we show $x_a(e_i) = x_*(e_i)$ $(i = 1, 2, \ldots, n)$ by induction on i. Assume that $x_a(e_h) = x_*(e_h)$ $(h = 1, 2, \ldots, i)$ holds for some $i < n$. For simplicity, we assume that $w(e_{i+1}) \geq 0$. By the definition of $x_a(e_{i+1})$, we have $x_a(e_{i+1}) \geq x_*(e_{i+1})$. On the other hand, it holds that $x_*(X_{i+1}, Y_{i+1}) = \rho(X_{i+1}, Y_{i+1}) \geq x_a(X_{i+1}, Y_{i+1})$, implying that

$$\begin{aligned} x_*(e_{i+1}) &= x_*(X_{i+1}, Y_{i+1}) - x_*(X_{i+1} \setminus \{e_{i+1}\}, Y_{i+1}) \\ &\geq x_a(X_{i+1}, Y_{i+1}) - x_a(X_{i+1} \setminus \{e_{i+1}\}, Y_{i+1}) \ = \ x_a(e_{i+1}). \end{aligned}$$

Hence, $x_a(e_{i+1}) = x_*(e_{i+1})$ holds. This shows that x_* coincides with the vector found by the greedy algorithm.

5.2 Relationship with Hartvigsen's Greedy Algorithm

We explain a greedy algorithm proposed by Hartvigsen [14] and discuss the relationship with our greedy algorithm.

Let $\mathcal{F} \subseteq \mathbb{Z}^E$ be an all-neighbor system. Recall that $D \subseteq \{0, +1, -1\}^E$ denotes the set of all directions (see Section 2); for every $x \in \mathcal{F}$ and every neighbor $y \in \mathcal{F}$ of x, we have $y - x = \alpha d$ for some $d \in D$ and $\alpha \in \mathbb{Z}_{++}$. For each direction $d \in D$, we define a value $w^p(d)$ by

$$w^p(d) = \frac{\sum_{e \in E} w(e)d(e)}{\sum_{e \in E} |d(e)|},$$

which means the "slope" of the direction d. The greedy algorithm of Hartvigsen [14] improves the current solution iteratively by using the steepest ascent direction, i.e., a direction d which maximizes the value $w^p(d)$ among all directions.

Hartvigsen's Greedy Algorithm for (Lin)
Step 0: Let x_0 be any vector in $\mathcal{F}$ and put $x := x_0$. Label all the directions $d_1, d_2, \ldots, d_m$ in D so that

$$w^p(d_1) \geq w^p(d_2) \geq \cdots \geq w^p(d_m).$$

Let m' be the maximum integer with $w^p(d_{m'}) \geq 0$.
Step 1: If there exists no d_j with $1 \leq j \leq m'$ and $\alpha \in \mathbb{Z}_{++}$ such that $x + \alpha d_j \in \mathcal{F}$, then output the current vector x and stop.
Step 2: For $j = 1, 2, \ldots, m'$, do the following: if $x + \alpha d_j \in \mathcal{F}$ holds for some $\alpha \in \mathbb{Z}_{++}$, then update x by $x := x + \alpha d_j$ and go to Step 1.

We note that the description of the greedy algorithm is slightly different from the original one in [14], which describes the algorithm by using a neighbor function.

A variant of Hartvigsen's greedy algorithm, called the *altered greedy algorithm* in [14], has a better time complexity. The altered greedy algorithm is the same as the original greedy algorithm of Hartvigsen, except that it replaces the coefficient vector w of the objective function to a new one, based on the following property:

Proposition 5.1 ([14, Proposition 13]). *Let $w, w_* \in \mathbb{R}^E$ be vectors satisfying the following conditions:*

$$\begin{aligned}
&\textit{for every } e \in E, \quad \textit{if } w(e) > 0 \textit{ then } w_*(e) > 0, \quad \textit{if } w(e) < 0 \textit{ then } w_*(e) < 0,\\
&\textit{for every } e \in E, \quad \textit{if } |w(e)| > 0 \textit{ then } |w_*(e)| > 0,\\
&\textit{for every } e, e' \in E, \quad \textit{if } |w(e)| > |w(e')| \textit{ then } |w_*(e)| \gg |w_*(e')|,
\end{aligned}$$

where $a \gg b$ means that a is sufficiently larger than b. Then, $\arg\max\{\sum_{e\in E} w_*(e)x(e) \mid x \in \mathcal{F}\}$ *is contained in* $\arg\max\{\sum_{e\in E} w(e)x(e) \mid x \in \mathcal{F}\}$.

A proof of Proposition 5.1 is given in [14]; an alternative proof can be given by using the fact that the convex closure of a neighbor system is a bisubmodular polyhedron (Theorem 4.2).

Given a coefficient vector $w \in \mathbb{R}^E$ of the problem (Lin), we consider a vector $w_* \in \mathbb{R}^E$ satisfying the conditions in Proposition 5.1. The altered greedy algorithm uses the vector w_* instead of w, and applies the greedy algorithm of Hartvigsen to the problem $\max\{\sum_{e\in E} w_*(e)x(e) \mid x \in \mathcal{F}\}$ to obtain an optimal solution of the original linear optimization problem.

We now explain the relationship between the altered greedy algorithm and our greedy algorithm. For the simplicity of the description, we assume, without loss of generality, that $E = \{e_1, e_2, \ldots, e_n\}$ and

$$w(e_1) > w(e_2) > \cdots > w(e_n) > 0.$$

Then, the vector $w_* \in \mathbb{R}^E$ satisfies

$$w_*(e_1) \gg w_*(e_2) \gg \cdots \gg w_*(e_n) \gg 0.$$

Therefore, the altered greedy algorithm firstly uses directions $d \in \{0, +1, -1\}^E$ with $d(e_1) = +1$, implying that the algorithm maximizes the value $x(e_1)$. The algorithm then uses directions

d with $d(e_1) = 0$ and $d(e_2) = +1$, implying that it maximizes the value $x(e_2)$ without changing the value $x(e_1)$. Similarly, the algorithm then maximizes the value $x(e_3)$ without changing the values $x(e_1)$ and $x(e_2)$, and so on. From this observation we see that the altered greedy algorithm is essentially equivalent to our greedy algorithm

We define the *size* of a set $\mathcal{F} \subseteq \mathbb{Z}^E$ as

$$\Phi(\mathcal{F}) = \max_{e \in E} \left\{ \max_{x \in \mathcal{F}} x(e) - \min_{x \in \mathcal{F}} x(e) \right\}; \tag{5.1}$$

$\Phi(\mathcal{F})$ is just the length of the longest edge in the bounding box of $\mathcal{F}$. It is shown in [14] that the altered greedy algorithm (and hence our greedy algorithm also) can be implemented to run in $\mathrm{O}(n^2\Phi(\mathcal{F}))$ time for a finite all-neighbor system and in $\mathrm{O}(kn^2 \log(\Phi(\mathcal{F})))$ time for a finite N_k-neighbor system, where it is assumed that we are given a membership oracle for $\mathcal{F}$, which enables us to check whether a given vector is contained in $\mathcal{F}$ or not in constant time. In particular, the analysis of the time complexity in [14] implies that the problems $\max\{x(e) \mid x \in \mathcal{F}\}$ and $\min\{x(e) \mid x \in \mathcal{F}\}$, which are special cases of (Lin), can be solved faster by a factor of n.

Theorem 5.2. *Let $e \in E$. For a finite all-neighbor system $\mathcal{F} \subseteq \mathbb{Z}^E$, the problems $\max\{x(e) \mid x \in \mathcal{F}\}$ and $\min\{x(e) \mid x \in \mathcal{F}\}$ can be solved in $\mathrm{O}(n\Phi(\mathcal{F}))$ time; if $\mathcal{F}$ is a finite N_k-neighbor system, then the problems can be solved in $\mathrm{O}(nk \log \Phi(\mathcal{F}))$ time.*

In the proof of Theorem 5.2 we use a fact that the (altered) greedy algorithm applied to $\max\{x(e) \mid x \in \mathcal{F}\}$ requires only directions $d \in D$ with $d(e) = +1$, and there exist at most $2n + 1$ such directions. Theorem 5.2 will be used in Section 6.

6 Separable Convex Optimization on Neighbor Systems

We consider the problem of minimizing a separable convex function on a finite neighbor system, which is formulated as follows:

$$\textbf{(SC)} \text{ Minimize } f(x) \equiv \sum_{e \in E} f_e(x(e)) \text{ subject to } x \in \mathcal{F},$$

where $f_e : \mathbb{Z} \to \mathbb{R}$ $(e \in E)$ is a family of univariate convex functions and $\mathcal{F} \subseteq \mathbb{Z}^E$ is a finite all-neighbor system. We propose efficient algorithms for (SC). We firstly show some useful properties in developing efficient algorithms for (SC) in Section 6.1. In Section 6.2, we propose a greedy algorithm for the problem (SC) and show that it runs in pseudo-polynomial time, i.e., time polynomial in $n = |E|$ and in the size $\Phi(\mathcal{F})$ of $\mathcal{F}$; recall the definition of $\Phi(\mathcal{F})$ in (5.1). We finally show in Section 6.3 that if $\mathcal{F}$ is an N_k-neighbor system, then the problem (SC) can be solved in weakly-polynomial time, i.e., time polynomial in n, in k, and in $\log \Phi(\mathcal{F})$.

Remark 6.1. As mentioned in Introduction, the separable convex optimization problem on *jump systems* can be solved in weakly-polynomial time by an algorithm. Hence, it is a natural idea to reduce the problem (SC) to an optimization problem on a jump system by using the relationship between neighbor systems and jump systems shown in Section 3, and then to apply the existing results on jump systems. Indeed, the problem (SC) can be reduced to the following optimization problem on a jump system $\mathcal{J} \subseteq \mathbb{Z}^E$:

$$\textbf{(SC}'\textbf{)} \qquad \text{Minimize } g(x) \equiv \sum_{e \in E} g_e(x(e)) \text{ subject to } x \in \mathcal{J},$$

where vectors $\ell, u \in \mathbb{Z}^E$ and a family of strictly increasing functions $\pi_e : [\ell(e), u(e)] \to \mathbb{Z}$ $(e \in E)$ are given as in Section 3, and functions $g_e : [\ell(e), u(e)] \to \mathbb{R}$ $(e \in E)$ are defined by

$$g_e(\alpha) = f_e(\pi_e(\alpha)) \qquad (\alpha \in [\ell(e), u(e)]).$$

This approach, however, does not lead to a polynomial-time algorithm for (SC) due to the following reasons.

One reason is that the objective function g of the problem (SC$'$) is not separable convex. In particular, each function g_e is quasi-convex but not convex in general, due to the nonlinear coordinate scaling π_e. Hence, algorithms for the separable convex optimization problem on jump systems cannot be applied to (SC$'$).

Another reason is that it is difficult to transformation a neighbor system $\mathcal{F}$ to a jump system $\mathcal{J}$ efficiently. Since $\mathcal{F}$ may contain an exponential number of vectors, it is better to compute functions π_e and obtain $\mathcal{J}$ implicitly, instead of computing $\mathcal{J}$ explicitly. We can indeed compute functions π_e by using the ideas in proofs of Section 3, but it still requires pseudo-polynomial time. □

6.1 Theorems

The next theorem shows that the optimality of a vector can be characterized by a local optimality. This is an extension of an optimality condition [3, Corollary 4.2] for jump systems.

Theorem 6.2. *Let $\mathcal{F} \subseteq \mathbb{Z}^E$ be an all-neighbor system, and $x \in \mathcal{F}$. Then, x is an optimal solution of* (SC) *if and only if $f(x) \le f(y)$ holds for all proper neighbor $y \in \mathcal{F}$ of x.*

Proof. The proof is given in Section 6.4.1. □

The next property, which is an extension of the corresponding result [23, Theorem 4.2] for jump systems, shows that a given nonoptimal vector in $\mathcal{F}$ can be easily separated from an optimal solution. Recall that U denotes the set of unit vectors given by (2.1).

For an all-neighbor system $\mathcal{F} \subseteq \mathbb{Z}^E$ and $x \in \mathcal{F}$, we define

$$\begin{aligned} \mathrm{PN}(\mathcal{F}, x) = \{(p, q, \alpha) \mid {} & p \in U,\ q \in U \cup \{\mathbf{0}\} \setminus \{+p, -p\},\ \alpha \in \mathbb{Z}_{++}, \\ & x + \alpha(p+q) \text{ is a proper neighbor of } x\}. \end{aligned} \tag{6.1}$$

We note that if $(p, q, \alpha) \in \mathrm{PN}(\mathcal{F}, x)$ and $q \neq \mathbf{0}$ then we also have $(q, p, \alpha) \in \mathrm{PN}(\mathcal{F}, x)$.

Theorem 6.3. *Let $\mathcal{F} \subseteq \mathbb{Z}^E$ be an all-neighbor system. Suppose that* (SC) *has an optimal solution, and let $x \in \mathcal{F}$ be a vector which is not an optimal solution of* (SC)*. Let (p_*, q_*, α_*) be an element in $\mathrm{PN}(\mathcal{F}, x)$ such that $f(x + \alpha_*(p_* + q_*)) < f(x)$, and assume that it minimizes the value $\{f(x + \alpha_* p_*) - f(x)\}/\alpha_*$ among all such elements. Then, there exists an optimal solution x_* of* (SC) *satisfying*

$$\begin{cases} x_*(i) \le x(i) - \alpha^- & (\textit{if } p_* = -\chi_i), \\ x_*(i) \ge x(i) + \alpha^+ & (\textit{if } p_* = +\chi_i), \end{cases}$$

where

$$\begin{aligned}\alpha^- &= \min\{x(i) - y(i) \mid y \in \mathcal{F},\ y(i) < x(i)\},\\ \alpha^+ &= \min\{y(i) - x(i) \mid y \in \mathcal{F},\ y(i) > x(i)\}.\end{aligned}$$

Proof. The proof is given in Section 6.4.2. □

We then show that the enumeration of proper neighbors of a given vector can be done efficiently. This property is useful in checking the local optimality in the sense of Theorem 6.2 and the computation of an element $(p_*, q_*, \alpha_*) \in \mathrm{PN}(\mathcal{F}, x)$ in Theorem 6.3.

Theorem 6.4. *Let $\mathcal{F} \subseteq \mathbb{Z}^E$ be an all-neighbor system and $x \in \mathcal{F}$. All proper neighbors of x can be computed in $\mathrm{O}(n^2\Phi(\mathcal{F}))$ time; it can be done in $\mathrm{O}(n^2k)$ time if $\mathcal{F}$ is an N_k-neighbor system.*

Proof. Since an all-neighbor system $\mathcal{F}$ is an N_k-neighbor system with $k = \Phi(\mathcal{F})$, it suffices to show the bound $\mathrm{O}(kn^2)$ for an N_k-neighbor system $\mathcal{F}$. In this case, computation of all proper neighbors of x can be done as follows. Recall that all proper neighbors are contained in $N_k(\mathcal{F}, x)$ by Proposition 2.3.

We firstly compute proper neighbors of the form $x + \alpha\chi_i$ $(i \in E,\ 0 < \alpha \le k)$. We set

$$\alpha_i^+ = \min\{\alpha \mid \alpha \in \mathbb{Z},\ x + \alpha\chi_i \in \mathcal{F},\ 0 < \alpha \le k\};$$

if $\{\alpha \in \mathbb{Z} \mid x + \alpha\chi_i \in \mathcal{F},\ 0 < \alpha \le k\} = \emptyset$, then set $\alpha_i^+ = +\infty$. If $\alpha_i^+ < +\infty$, then output $x + \alpha_i^+\chi_i$, which is a proper neighbor of x; otherwise, there exists no proper neighbor of the form $x + \alpha\chi_i$. It is easy to see that this can be done in $\mathrm{O}(nk)$ time for all $i \in E$. Similarly, we can compute all proper neighbors of the form $x - \alpha\chi_i$ $(i \in E,\ 0 < \alpha \le k)$ in $\mathrm{O}(nk)$ time.

We then compute proper neighbors of the form $x + \alpha(\chi_i + \chi_j)$ for some distinct $i, j \in E$ and $\alpha \in \mathbb{Z}$ with $0 < \alpha \le k$. If the set $\{\alpha \mid x + \alpha(\chi_i + \chi_j) \in \mathcal{F},\ 0 < \alpha \le k\}$ is nonempty, then we compute the value α_{ij}^+ defined by

$$\alpha_{ij}^+ = \min\{\alpha \mid x + \alpha(\chi_i + \chi_j) \in \mathcal{F},\ 0 < \alpha \le k\}.$$

If $\alpha_{ij}^+ < \alpha_i^+$ or $\alpha_{ij}^+ < \alpha_j^+$, then the vector $x + \alpha_{ij}^+(\chi_i + \chi_j)$ is a proper neighbor, and we output it. It is easy to see that this can be done in $\mathrm{O}(n^2k)$ time. Similarly, we can compute all proper neighbors of the forms $x + \alpha(\chi_i - \chi_j)$ and $x + \alpha(-\chi_i - \chi_j)$ in $\mathrm{O}(n^2k)$ time. Hence, we can compute all proper neighbors of x in $\mathrm{O}(n^2k)$ time. □

6.2 Pseudopolynomial-Time Algorithm

Based on Theorems 6.2 and 6.3, we propose a greedy algorithm for solving the problem (SC). We assume in this section that $\mathcal{F}$ is a finite all-neighbor system, unless otherwise stated. The greedy algorithm maintains an interval $[a, b]$ with $a, b \in \mathbb{Z}^E$ containing an optimal solution of (SC). Note that $\mathcal{F} \cap [a, b]$ is a neighbor system by Proposition 2.1 (iii). The vectors a and b are updated by using Theorem 6.3 so that the value $||b - a||_1$ reduces in each iteration, and finally an optimal solution is found. We assume that an initial vector $x_0 \in \mathcal{F}$ is given.

Algorithm GREEDY_SC
Step 0: Let $x := x_0 \in \mathcal{F}$. Set $a(e) := a_{\mathcal{F}}(e)$ and $b(e) := b_{\mathcal{F}}(e)$, where

$$a_{\mathcal{F}}(e) = \min\{x(e) \mid x \in \mathcal{F}\}, \quad b_{\mathcal{F}}(e) = \max\{x(e) \mid x \in \mathcal{F}\} \qquad (e \in E). \tag{6.2}$$

Step 1: If $f(x) \leq f(y)$ holds for all proper neighbors y of x in $\mathcal{F} \cap [a, b]$, then output the current x and stop.
Step 2: Let (p_*, q_*, α_*) be an element in $\mathrm{PN}(\mathcal{F}, x)$ with $f(x + \alpha_*(p_* + q_*)) < f(x)$ minimizing the value $\{f(x + \alpha_* p_*) - f(x)\}/\alpha_*$ among all such elements.
Step 3: Modify a or b as follows:

$$\begin{cases} b(i) := x(i) - \alpha^- & (\text{if } p_* = -\chi_i), \\ a(i) := x(i) + \alpha^+ & (\text{if } p_* = +\chi_i), \end{cases} \tag{6.3}$$

where α^-, α^+ are defined by

$$\begin{cases} \alpha^- &= \min\{x(i) - y(i) \mid y \in \mathcal{F} \cap [a, b],\ y(i) < x(i)\}, \\ \alpha^+ &= \min\{y(i) - x(i) \mid y \in \mathcal{F} \cap [a, b],\ y(i) > x(i)\}. \end{cases} \tag{6.4}$$

Set $x := x'$. Go to Step 1. □

By Theorem 6.2, the output x of the algorithm is a minimizer of the function f in the set $\mathcal{F} \cap [a, b]$. We see from Theorem 6.3 that the set $\mathcal{F} \cap [a, b]$ always contains an optimal solution of (SC). Hence, the output x of the algorithm is an optimal solution of (SC).

We analyze the running time of the algorithm GREEDY_SC. By Theorem 5.2, the values $a_{\mathcal{F}}(e)$ and $b_{\mathcal{F}}(e)$ can be computed in $\mathrm{O}(n\Phi(\mathcal{F}))$ time for each $e \in E$. Hence, Step 0 can be done in $\mathrm{O}(n^2\Phi(\mathcal{F}))$ time. Steps 1 and 2 can be done in $\mathrm{O}(n^2\Phi(\mathcal{F}))$ time by Theorem 6.4. The values α^-, α^+ can be also computed in $\mathrm{O}(n\Phi(\mathcal{F}))$ time by Theorem 5.2 since the sets $\{y \in \mathcal{F} \cap [a, b] \mid y(i) < x(i)\}$ and $\{y \in \mathcal{F} \cap [a, b] \mid y(i) > x(i)\}$ are all-neighbor systems if they are nonempty. Hence, each iteration of the algorithm requires $\mathrm{O}(n^2\Phi(\mathcal{F}))$ time.

To bound the number of iterations of the algorithm, we use the value $||b - a||_1$. Suppose that we have $p_* = -\chi_i$ in Step 3, and denote by b_{old} (resp., b_{new}) the vector b before update (resp., after update). Then, it holds that $a(i) \leq b_{\mathrm{new}}(i) = x(i) - \alpha^- < x(i) \leq b_{\mathrm{old}}(i)$, implying that $||b_{\mathrm{new}} - a||_1 < ||b_{\mathrm{old}} - a||_1$. If $p_* = +\chi_i$ holds in Step 3, then we can show in the same way that $||b - a_{\mathrm{new}}||_1 < ||b - a_{\mathrm{old}}||_1$, where a_{old} (resp., a_{new}) the vector a before update (resp., after update). Hence, the value $||b - a||_1$ reduces in each iteration, and therefore the number of iterations is bounded by $n\,\Phi(\mathcal{F})$.

Theorem 6.5. *The algorithm* GREEDY_SC *finds an optimal solution of the problem* (SC) *in* $\mathrm{O}(n^3\,\Phi(\mathcal{F})^2)$ *time.*

6.3 Polynomial-Time Algorithm

We propose a faster algorithm for (SC) with a finite N_k-neighbor system $\mathcal{F}$ by using the domain reduction approach. The domain reduction approach has been used to develop polynomial-time algorithms for various optimization problems with discrete convex objective functions [22, 23]. We show that the proposed algorithm runs in weakly polynomial time if $\mathcal{F}$ is an N_k-neighbor system and the value k is known a priori.

Given a finite N_k-neighbor system $\mathcal{F} \subseteq \mathbb{Z}^E$, we define a set $\mathcal{F}^\bullet \subseteq \mathbb{Z}^E$ by $\mathcal{F}^\bullet = \mathcal{F} \cap [a^\bullet_{\mathcal{F}}, b^\bullet_{\mathcal{F}}]$, where $a_{\mathcal{F}}, b_{\mathcal{F}} \in \mathbb{Z}^E$ are defined by (6.2) and

$$a^\bullet_{\mathcal{F}}(e) = a_{\mathcal{F}}(e) + \left\lfloor \frac{b_{\mathcal{F}}(e) - a_{\mathcal{F}}(e)}{nk} \right\rfloor, \quad b^\bullet_{\mathcal{F}}(e) = b_{\mathcal{F}}(e) - \left\lfloor \frac{b_{\mathcal{F}}(e) - a_{\mathcal{F}}(e)}{nk} \right\rfloor \quad (e \in E).$$

Theorem 6.6. *Let $\mathcal{F} \subseteq \mathbb{Z}^E$ be a finite N_k-neighbor system.*
(i) *The set $\mathcal{F}^\bullet$ is nonempty and hence an N_k-neighbor system.*
(ii) *A vector in $\mathcal{F}^\bullet$ can be found in $\mathrm{O}(n^2 k \log \Phi(\mathcal{F}))$ time, provided a vector in $\mathcal{F}$ is given.*

Proof. Proofs of (i) and (ii) are given in Sections 6.4.3 and 6.4.4, respectively. □

The algorithm is described as follows. Assume that an initial vector $x_0 \in \mathcal{F}$ is given.

Algorithm Domain_Reduction
Step 0: Set $a := a_{\mathcal{F}}$ and $b := b_{\mathcal{F}}$, where $a_{\mathcal{F}}$ and $b_{\mathcal{F}}$ are given by (6.2).
Step 1: Find a vector $x \in (\mathcal{F} \cap [a, b])^\bullet$.
Step 2: If $f(x) \le f(y)$ holds for all proper neighbors y of x in $\mathcal{F} \cap [a, b]$, then output the current x and stop.
Step 3: Let (p_*, q_*, α_*) be an element in $\mathrm{PN}(\mathcal{F} \cap [a, b], x)$ with $f(x + \alpha_*(p_* + q_*)) < f(x)$ minimizing the value $\{f(x + \alpha_* p_*) - f(x)\}/\alpha_*$ among all such elements.
Step 4: Modify a or b as in Step 3 of Greedy_SC. Go to Step 1. □

The validity of this algorithm can be shown in a similar way as the algorithm Greedy_SC by using Theorems 6.2 and 6.3.

We analyze the number of iterations. Denote by a_m, b_m the vectors a, b in Step 1 of the m-th iteration, and define vectors $a'_m, b'_m \in \mathbb{Z}^E$ by

$$a'_m(e) = \min\{x(e) \mid x \in \mathcal{F} \cap [a_m, b_m]\}, \quad b'_m(e) = \max\{x(e) \mid x \in \mathcal{F} \cap [a_m, b_m]\}.$$

It is noted that $a'_m(e) \ge a_m(e)$ and $b'_m(e) \le b_m(e)$ hold for every $e \in E$ and $\mathcal{F} \cap [a'_m, b'_m] = \mathcal{F} \cap [a_m, b_m]$ holds. In addition, it is clear that the value $a'_m(e)$ is nondecreasing and $b'_m(e)$ is nonincreasing with respect to m for each $e \in E$. The following property is the key to obtain a polynomial bound.

Lemma 6.7. *Let p_* be the vector chosen in Step 3 of the m-th iteration, and $i \in E$ be the element with $\{i\} = \mathrm{supp}(p_*)$. Then, it holds that*

$$b'_{m+1}(i) - a'_{m+1}(i) < \left(1 - \frac{1}{nk}\right) \{b'_m(i) - a'_m(i)\}.$$

Proof. We show the inequality in the case $p_* = -\chi_i$ only since the case $p_* = +\chi_i$ can be shown similarly. Let $x \in (\mathcal{F} \cap [a_m, b_m])^\bullet$ be the vector chosen in Step 1 of the m-th iteration. By using the inequalities $b'_{m+1}(i) \le b_{m+1}(i)$ and $a'_{m+1}(i) \ge a'_m(i)$, we have

$$b'_{m+1}(i) - a'_{m+1}(i) \le b_{m+1}(i) - a'_m(i). \tag{6.5}$$

By (6.3), it holds that

$$b_{m+1}(i) = x(u) - \alpha^-. \tag{6.6}$$

Since $x \in (\mathcal{F} \cap [a_m, b_m])^\bullet = (\mathcal{F} \cap [a'_m, b'_m])^\bullet$, we have

$$x(i) \leq b'_m(i) - \left\lfloor \frac{b'_m(i) - a'_m(i)}{nk} \right\rfloor. \tag{6.7}$$

From (6.5), (6.6), and (6.7) follows that

$$\begin{aligned} b'_{m+1}(i) - a'_{m+1}(i) &\leq \left(b'_m(i) - \left\lfloor \frac{b'_m(i) - a'_m(i)}{nk} \right\rfloor \right) - \alpha^- - a'_m(i) \\ &\leq \left(b'_m(i) - \left\lfloor \frac{b'_m(i) - a'_m(i)}{nk} \right\rfloor \right) - 1 - a'_m(i) \\ &< \left(1 - \frac{1}{nk} \right) \{ b'_m(i) - a'_m(i) \}. \end{aligned}$$

□

We have $b'_1(e) - a'_1(e) = b_1(e) - a_1(e) = b_\mathcal{F}(e) - a_\mathcal{F}(e) \leq \Phi(\mathcal{F})$ for all $e \in E$ at the beginning of the algorithm, and if $b'_m(e) - a'_m(e) < 1$ for all $e \in E$, then we obtain an optimal solution immediately. Hence, it follows from Lemma 6.7 that the algorithm DOMAIN_REDUCTION terminates in $\mathrm{O}(n^2 k \log \Phi(\mathcal{F}))$ iterations.

Recall that $\mathcal{F}_k$ is an N_k-neighbor system. By Theorem 5.2, the values $a_\mathcal{F}(e)$ and $b_\mathcal{F}(e)$ can be computed in $\mathrm{O}(nk \log \Phi(\mathcal{F}))$ time for each $e \in E$. Hence, Step 0 can be done in $\mathrm{O}(n^2 k \log \Phi(\mathcal{F}))$ time. By Theorem 6.6 (ii), Step 1 can be done in $\mathrm{O}(n^2 k \log \Phi(\mathcal{F}))$ time. Step 2 can be done in $\mathrm{O}(n^2 k)$ time by Theorem 6.4. The values α^-, α^+ in Step 3 can be also computed in $\mathrm{O}(n^2 k)$ time by the following property:

Proposition 6.8. *Let $\mathcal{F} \subseteq \mathbb{Z}^E$ be an all-neighbor system and $x \in \mathcal{F}$.*
(i) *Suppose that $\{y \in \mathcal{F} \mid y(i) < x(i)\} \neq \emptyset$. Then, there exists a proper neighbor $y_* \in \mathcal{F}$ of x such that $x(i) - y_*(i) = \min\{x(i) - y(i) \mid y \in \mathcal{F},\ y(i) < x(i)\}$.*
(ii) *Suppose that $\{y \in \mathcal{F} \mid y(i) > x(i)\} \neq \emptyset$. Then, there exists a proper neighbor $y_* \in \mathcal{F}$ of x such that $y_*(i) - x(i) = \min\{y(i) - x(i) \mid y \in \mathcal{F},\ y(i) > x(i)\}$.*

Proof. The statements (i) and (ii) follow from the discussion in Section 3.2.1. □

Note that the sets $\{y \in \mathcal{F} \cap [a, b] \mid y(i) < x(i)\}$ and $\{y \in \mathcal{F} \cap [a, b] \mid y(i) > x(i)\}$ are N_k-neighbor systems if they are nonempty. Hence, each iteration of the algorithm runs in $\mathrm{O}(n^2 k \log \Phi(\mathcal{F}))$ time.

Theorem 6.9. *The algorithm* DOMAIN_REDUCTION *finds an optimal solution of the problem* (SC) *in $\mathrm{O}(n^4 k^2 (\log \Phi(\mathcal{F}))^2)$ time if $\mathcal{F}$ is a finite N_k-neighbor system.*

6.4 Proofs

In the proofs below, we often use the following fundamental properties of separable convex functions.

Proposition 6.10. *Let $f : \mathbb{Z}^E \to \mathbb{R}$ be a separable convex function.*
(i) *Let $x, y \in \mathbb{Z}^E$ and $p \in \mathrm{inc}(x, y)$. For every $\alpha, \beta \in \mathbb{Z}_{++}$, if both of $x + \alpha p$ and $y - \beta p$ are between x and y, then we have*

$$\frac{f(x + \alpha p) - f(x)}{\alpha} \leq \frac{f(y) - f(y - \beta p)}{\beta}.$$

(ii) *Let* $x \in \mathbb{Z}^E$, $p \in U$, $q \in U \cup \{\mathbf{0}\} \setminus \{+p, -p\}$, *and* $\alpha, \beta \in \mathbb{Z}_{++}$. *Then, we have*

$$f(x + \alpha p + \beta q) - f(x) = \{f(x + \alpha p) - f(x)\} + \{f(x + \beta q) - f(x)\}.$$

6.4.1 Proof of Theorem 6.2

The "only if" part is obvious. We give a proof of "if" part below.

We firstly consider the case where $\mathcal{F}$ is a finite set. This implies the existence of an optimal solution of (SC), in particular. Let $x_* \in \mathcal{F}$ be an optimal solution of (SC) minimizing the value $\|x_* - x\|_1$. If $\|x_* - x\|_1 = 0$, then $x_* = x$ and we are done. Hence, we assume, to the contrary, that $x_* \neq x$ and derive a contradiction.

Since $x_* \neq x$, there exists some $(p, q, \alpha) \in \mathrm{PN}(\mathcal{F}, x)$ such that $x + \alpha(p + q)$ is between x and x_*, and suppose that (p, q, α) maximizes the value $\{f(x + \alpha p) - f(x)\}/\alpha$ among all such elements in $\mathrm{PN}(\mathcal{F}, x)$; recall the definition of $\mathrm{PN}(\mathcal{F}, x)$ in (6.1).

We show that

$$f(x + \alpha p) - f(x) \geq 0. \tag{6.8}$$

Since $x + \alpha(p + q)$ is a proper neighbor of x, we have $f(x + \alpha(p + q)) \geq f(x)$. Hence, (6.8) holds immediately from this inequality if $q = \mathbf{0}$. If $q \neq \mathbf{0}$, then Proposition 6.10 (ii) implies that

$$\begin{aligned} 0 &\leq \frac{f(x + \alpha(p + q)) - f(x)}{\alpha} \\ &= \frac{f(x + \alpha p) - f(x)}{\alpha} + \frac{f(x + \alpha q) - f(x)}{\alpha} \leq 2 \cdot \frac{f(x + \alpha p) - f(x)}{\alpha}, \end{aligned}$$

where the last inequality is by the choice of (p, q, α). Hence, (6.8) holds.

Since $-p \in \mathrm{inc}(x_*, x)$, the property (NS$'$) and Theorem 2.2 imply that there exist $-s \in \mathrm{inc}(x_*, x) \cup \{\mathbf{0}\} \setminus \{+p, -p\}$ and $\beta \in \mathbb{Z}_{++}$ such that $x_* - \beta(p + s)$ is a proper neighbor of x_* between x_* and x. Since $x + \alpha p$ and $x_* - \beta p$ are between x and x_*, Proposition 6.10 (i) implies that

$$\frac{f(x_*) - f(x_* - \beta p)}{\beta} \geq \frac{f(x + \alpha p) - f(x)}{\alpha} \geq 0. \tag{6.9}$$

Hence, we have $f(x_* - \beta p) \leq f(x_*)$. In addition, we have $f(x_* - \beta(p + s)) > f(x_*)$ since $\|(x_* - \beta(p + s)) - x\|_1 < \|x_* - x\|_1$. Therefore, we have $s \neq \mathbf{0}$. It follows from Proposition 6.10 (ii) that

$$0 < f(x_* - \beta(p + s)) - f(x_*) = \{f(x_* - \beta p) - f(x_*)\} + \{f(x_* - \beta s) - f(x_*)\},$$

implying that

$$\frac{f(x_* - \beta s) - f(x_*)}{\beta} > \frac{f(x_*) - f(x_* - \beta p)}{\beta}. \tag{6.10}$$

Since $s \in \mathrm{inc}(x, x_*)$, the property (NS$'$) and Theorem 2.2 imply that there exist $t \in \mathrm{inc}(x, x_*) \cup \{\mathbf{0}\} \setminus \{+s, -s\}$ and $\gamma \in \mathbb{Z}_{++}$ such that $x + \gamma(s + t)$ is a proper neighbor of x between x and x_*. Since $x_* - \beta s$ and $x + \gamma s$ are between x_* and x, Proposition 6.10 (i) implies that

$$\frac{f(x) - f(x + \gamma s)}{\gamma} \geq \frac{f(x_* - \beta s) - f(x_*)}{\beta}. \tag{6.11}$$

By combining (6.9), (6.10), and (6.11), we have $\{f(x) - f(x + \gamma s)\}/\gamma > 0$. Hence, it holds that $f(x + \gamma s) < f(x)$, which, combined with the choice of x, implies that $t \neq \mathbf{0}$. In addition, we have $f(x + \gamma(s + t)) \geq f(x)$ by the choice of x. Therefore, it follows from Proposition 6.10 (ii) that

$$0 \leq f(x + \gamma(s + t)) - f(x) = \{f(x + \gamma s) - f(x)\} + \{f(x + \gamma t) - f(x)\},$$

implying that

$$\frac{f(x + \gamma t) - f(x)}{\gamma} \geq \frac{f(x) - f(x + \gamma s)}{\gamma}.$$

This inequality, together with (6.9), (6.10), and (6.11), yields

$$\frac{f(x + \gamma t) - f(x)}{\gamma} > \frac{f(x + \alpha p) - f(x)}{\alpha},$$

a contradiction to the choice of (p, q, α). This concludes the proof for the optimality of x when $\mathcal{F}$ is a finite set.

We then consider the case where $\mathcal{F}$ is an infinite set. For every positive integer α, we define a set $\mathcal{F}_\alpha = \{y \in \mathcal{F} \mid \|y - x\|_\infty \leq \alpha\}$. Then, $\mathcal{F}_\alpha$ is an all-neighbor system by Proposition 2.1 (iii). Therefore, the discussion above implies that the vector x minimizes the function value of f among all vectors in $\mathcal{F}_\alpha$. Since this holds for all nonnegative integer α, x minimizes the function value of f among all vectors in $\mathcal{F}$, i.e., x is an optimal solution of (SC).

6.4.2 Proof of Theorem 6.3

The proof given below is similar to that for [23, Theorem 4.2].

Let $\mathcal{F} \subseteq \mathbb{Z}^E$ be an all-neighbor system, and $x \in \mathcal{F}$ be a vector which is not an optimal solution of (SC). Then, Theorem 6.2 implies that there exists some element $(p_*, q_*, \alpha_*) \in \mathrm{PN}(\mathcal{F}, x)$ such that

$$f(x + \alpha_*(p_* + q_*)) < f(x). \tag{6.12}$$

We assume that (p_*, q_*, α_*) minimizes the value $\{f(x + \alpha_* p_*) - f(x)\}/\alpha_*$ among all such elements. Assume, without loss of generality, that $p_* = +\chi_i$ for some $i \in E$. Let x_* be an optimal solution of (SC) maximizing the value $x_*(i)$, and assume that x_* minimizes $\|x_* - x\|_1$ among all such optimal solutions. We assume, to the contrary, that $x_*(i) \leq x(i)$ and derive a contradiction.

Lemma 6.11. $f(x + \alpha_* p_*) < f(x)$.

Proof. If $q_* = \mathbf{0}$, then the claim follows immediately from (6.12). Hence, we assume $q_* \neq \mathbf{0}$. Then, it holds that

$$\begin{aligned} 2 \cdot \frac{f(x + \alpha_* p_*) - f(x)}{\alpha_*} &\leq \frac{f(x + \alpha_* p_*) - f(x)}{\alpha_*} + \frac{f(x + \alpha_* q_*) - f(x)}{\alpha_*} \\ &= \frac{f(x + \alpha_*(p_* + q_*)) - f(x)}{\alpha_*} < 0, \end{aligned}$$

where the first inequality is by the choice of (p_*, q_*, α_*), the equation is by Proposition 6.10 (ii), and the last inequality is by (6.12). $\square$

The next lemma shows a key property in the proof of Theorem 6.3.

Lemma 6.12. *There exists some* $(-s, t, \beta) \in \mathrm{PN}(\mathcal{F}, x)$ *such that* $s \in \mathrm{inc}(x_*, x)$ *and*

$$\frac{f(x - \beta s) - f(x)}{\beta} < \frac{f(x + \alpha_* p_*) - f(x)}{\alpha_*}. \tag{6.13}$$

Proof. Since $p_* = +\chi_i$ and $x_*(i) \leq x(i)$, we have $p_* \in \mathrm{inc}(x_*, x + \alpha_*(p_* + q_*))$. Therefore, the property (NS$'$) and Theorem 2.2 imply that there exist some $s \in \mathrm{inc}(x_*, x+\alpha_*(p_*+q_*)) \cup \{\mathbf{0}\} \setminus \{+p_*, -p_*\}$ and $\gamma \in \mathbb{Z}_{++}$ such that $x_* + \gamma(p_* + s)$ is a proper neighbor of x_* between x_* and $x + \alpha_*(p_* + q_*)$.

We show that $s \neq \mathbf{0}$. Recall that x_* is an optimal solution of (SC) with the maximum value of $x_*(i)$. Hence, it holds that $f(x_* + \gamma(p_* + s)) > f(x_*)$, which, together with Proposition 6.10 (ii), implies that

$$\frac{f(x_* + \gamma p_*) - f(x_*)}{\gamma} + \frac{f(x_* + \gamma s) - f(x_*)}{\gamma} = \frac{f(x_* + \gamma(p_* + s)) - f(x_*)}{\gamma} > 0. \tag{6.14}$$

Since $x_* + \gamma p_*$ and x are between x_* and $x + \alpha_* p_*$, it follows from Proposition 6.10 (i) and Lemma 6.11 that

$$\frac{f(x_* + \gamma p_*) - f(x_*)}{\gamma} \leq \frac{f(x + \alpha_* p_*) - f(x)}{\alpha_*} < 0. \tag{6.15}$$

If $s = \mathbf{0}$, the inequality (6.14) implies $f(x_* + \gamma p_*) - f(x_*) > 0$, a contradiction to (6.15). Hence, we have $s \neq \mathbf{0}$.

We then prove $s \in \mathrm{inc}(x_*, x)$. Suppose, to the contrary, that $s \notin \mathrm{inc}(x_*, x)$. Since $s \neq \mathbf{0}$, we have $s = q_* \neq \mathbf{0}$. We may assume, without loss of generality, that $q_* = +\chi_j$ for some $j \in E \setminus \{i\}$. Since $s = +\chi_j \notin \mathrm{inc}(x_*, x)$, we have $x_*(j) \geq x(j)$. If $x_*(j) > x(j)$, then Lemma 3.3 implies that $x_*(j) \geq x(j) + \alpha_*$, a contradiction to the fact that $x_* + \gamma(p_* + q_*)$ is between x_* and $x + \alpha_*(p_* + q_*)$. Hence, we have $x_*(j) = x(j)$. Then, it follows from Lemma 3.3 that $\gamma = \alpha_*$, implying that

$$\frac{f(x_* + \gamma q_*) - f(x_*)}{\gamma} = \frac{f(x + \alpha_* q_*) - f(x)}{\alpha_*}. \tag{6.16}$$

By (6.14), (6.15), (6.16), and Proposition 6.10 (ii), it holds that

$$0 < \frac{f(x + \alpha_* p_*) - f(x)}{\alpha_*} + \frac{f(x + \alpha_* q_*) - f(x)}{\alpha_*} = \frac{f(x + \alpha_*(p_* + q_*)) - f(x)}{\alpha_*},$$

a contradiction to the inequality (6.12). Hence, we have $s \in \mathrm{inc}(x_*, x)$.

Since $-s \in \mathrm{inc}(x, x_*)$, the property (NS$'$) and Theorem 2.2 imply that there exist some $t \in \mathrm{inc}(x, x_*) \cup \{\mathbf{0}\} \setminus \{+s, -s\}$ and $\beta \in \mathbb{Z}_{++}$ such that $x + \beta(-s + t)$ is a proper neighbor of x between x and x_*. We note that $x_* + \gamma(p_* + s)$ is between x_* and x since it is between x_* and $x + \alpha_*(p_* + q_*)$ and $s \notin \{p_*, q_*\}$. Hence, Proposition 6.10 (i) implies that

$$\frac{f(x_* + \gamma s) - f(x_*)}{\gamma} \leq \frac{f(x) - f(x - \beta s)}{\beta}. \tag{6.17}$$

From (6.14), (6.15), and (6.17) follows (6.13). $\square$

Let $(-s, t, \beta)$ be an element in $\mathrm{PN}(\mathcal{F}, x)$ satisfying $s \in \mathrm{inc}(x_*, x)$, and assume that it minimizes the value $\{f(x - \beta s) - f(x)\}/\beta$ among all such elements in $\mathrm{PN}(\mathcal{F}, x)$. By Lemma 6.12, we have

$$\frac{f(x - \beta s) - f(x)}{\beta} < \frac{f(x + \alpha_* p_*) - f(x)}{\alpha_*}. \tag{6.18}$$

By the choice of (p_*, q_*, α_*), we have $f(x + \beta(-s + t)) \geq f(x)$. By (6.18) and Lemma 6.11, it holds that $f(x - \beta s) < f(x)$, implying that $t \neq \mathbf{0}$. By $t \neq \mathbf{0}$, $f(x + \beta(-s + t)) \geq f(x)$, and Proposition 6.10 (ii), it holds that

$$\frac{f(x + \beta t) - f(x)}{\beta} \geq \frac{f(x) - f(x - \beta s)}{\beta} > 0. \tag{6.19}$$

Since $-t \in \mathrm{inc}(x_*, x)$, there exist some $s' \in \mathrm{inc}(x_*, x) \cup \{\mathbf{0}\} \setminus \{+t, -t\}$ and $\eta \in \mathbb{Z}_{++}$ such that $x' \equiv x_* + \eta(-t + s')$ is a proper neighbor of x_* between x_* and x. By Proposition 6.10 (i), it holds that

$$\frac{f(x + \beta t) - f(x)}{\beta} \leq \frac{f(x_*) - f(x_* - \eta t)}{\eta}. \tag{6.20}$$

Since $-t, s' \in \mathrm{inc}(x_*, x)$ and $x_*(i) \leq x(i)$, we have $x'(i) \geq x_*(i)$, which, combined with $\|(x_* + \eta(-t + s')) - x\|_1 < \|x_* - x\|_1$, implies $f(x_* + \eta(-t + s')) > f(x_*)$. From (6.19) and (6.20) follows $f(x_*) > f(x_* - \eta t)$. Hence, $s' \neq \mathbf{0}$ holds. By $f(x_* + \eta(-t + s')) > f(x_*)$ and Proposition 6.10 (ii), we have

$$\frac{f(x_* - \eta t) - f(x_*)}{\eta} + \frac{f(x_* + \eta s') - f(x_*)}{\eta} > 0. \tag{6.21}$$

Since $-s' \in \mathrm{inc}(x, x_*)$, there exist some $t' \in \mathrm{inc}(x, x_*) \cup \{\mathbf{0}\} \setminus \{+s', -s'\}$ and $\xi \in \mathbb{Z}_{++}$ such that $x + \xi(-s' + t')$ is a proper neighbor of x between x and x_*. By Proposition 6.10 (i), it holds that

$$\frac{f(x_* + \eta s') - f(x_*)}{\eta} \leq \frac{f(x) - f(x - \xi s')}{\xi}. \tag{6.22}$$

By combining (6.19), (6.20), (6.21), and (6.22), we obtain

$$\frac{f(x - \xi s') - f(x)}{\xi} < \frac{f(x - \beta s) - f(x)}{\beta},$$

a contradiction to the choice of $(-s, t, \beta)$.

6.4.3 Proof of Theorem 6.6 (i)

To prove the statement (i), we use the following property of jump systems.

Theorem 6.13 ([23, Theorem 4.3]). *Let $\mathcal{J} \subseteq \mathbb{Z}^E$ be a jump system. Define a set $\mathcal{J}^\circ \subseteq \mathbb{Z}^E$ by $\mathcal{J}^\circ = \mathcal{J} \cap [a^\circ_{\mathcal{J}}, b^\circ_{\mathcal{J}}]$, where for each $e \in E$,*

$$\begin{aligned} a_{\mathcal{J}}(e) &= \min\{x(e) \mid x \in \mathcal{J}\}, & b_{\mathcal{J}}(e) &= \max\{x(e) \mid x \in \mathcal{J}\}, \\ a^\circ_{\mathcal{J}}(e) &= a_{\mathcal{J}}(e) + \left\lfloor \frac{b_{\mathcal{J}}(e) - a_{\mathcal{J}}(e)}{n} \right\rfloor, & b^\circ_{\mathcal{J}}(e) &= b_{\mathcal{J}}(e) - \left\lfloor \frac{b_{\mathcal{J}}(e) - a_{\mathcal{J}}(e)}{n} \right\rfloor. \end{aligned}$$

Then, the set $\mathcal{J}^\circ$ is nonempty.

Let $\mathcal{F}$ be a given N_k-neighbor system, and define vectors $\ell, u \in \mathbb{Z}^E$, a jump system $\mathcal{J} \subseteq \mathbb{Z}^E$, and a family of strictly increasing functions $\pi_e : [\ell(e), u(e)] \to \mathbb{Z}$ $(e \in E)$ as in Section 3. For $e \in E$, let $\ell'(e)$ be the minimum integer with $\pi_e(\ell'(e)) \geq a^\bullet_{\mathcal{F}}(e)$ and $u'(e)$ be the maximum integer with $\pi_e(u'(e)) \leq b^\bullet_{\mathcal{F}}(e)$. Then, we have

$$\mathcal{F}^\bullet = \{x \in \mathcal{F} \mid \pi_e(\ell'(e)) \leq x(e) \leq \pi_e(u'(e)) \ (\forall e \in E)\}.$$

Therefore, $\mathcal{F}^\bullet$ is nonempty if and only if the set $\mathcal{J}' \subseteq \mathcal{J}$ given by

$$\mathcal{J}' = \{x \in \mathcal{J} \mid \ell'(e) \le x(e) \le u'(e) \ (\forall e \in E)\}$$

is nonempty. By Theorem 6.13, the set $\mathcal{J}'$ is nonempty if $\mathcal{J}' \supseteq \mathcal{J}^\circ$, i.e., if it holds that

$$\ell'(e) \le \ell(e) + \left\lfloor \frac{u(e)-\ell(e)}{n} \right\rfloor, \quad u'(e) \ge u(e) - \left\lfloor \frac{u(e)-\ell(e)}{n} \right\rfloor \qquad (\forall e \in E). \tag{6.23}$$

In the following, we prove the former inequality in (6.23) only since the latter can be proven in a similar way. For $e \in E$, it holds that

$$(\ell'(e)-1) - \ell(e) \le \pi_e(\ell'(e)-1) - \pi_e(\ell(e)) \le (a^\bullet_{\mathcal{F}}(e)-1) - a_{\mathcal{F}}(e),$$

where the first inequality follows from the fact that π_e is a strictly increasing, integer-valued function, and the second is by the definition of $\ell'(e)$. Hence, we have

$$\ell'(e) - \ell(e) \le a^\bullet_{\mathcal{F}}(e) - a_{\mathcal{F}}(e) = \left\lfloor \frac{b_{\mathcal{F}}(e) - a_{\mathcal{F}}(e)}{nk} \right\rfloor. \tag{6.24}$$

Lemma 6.14. *It holds that* $\pi_e(\alpha+1) - \pi_e(\alpha) \le k$ $(\forall e \in E,\ \ell(e) \le \forall \alpha < u(e))$.

Proof. Let $x \in \mathcal{F}$ (resp., $y \in \mathcal{F}$) be a vector with $x(e) = \pi_e(\alpha)$ (resp., $y(e) = \pi_e(\alpha+1)$). Since $x(e) < y(e)$, the property (NS) implies that there exist a neighbor x' of x such that $x(e) < x'(e) \le y(e)$, and $||x' - x||_1 \le k$. By the definition of the value $\pi_e(\alpha+1)$, we have $x'(e) = \pi_e(\alpha+1) = y(e)$. Hence, $\pi_e(\alpha+1) - \pi_e(\alpha) = y(e) - x(e) \le ||x'-x||_1 \le k$. $\square$

It follows from Lemma 6.14 that

$$b_{\mathcal{F}}(e) - a_{\mathcal{F}}(e) \le k(u(e) - \ell(e)) \qquad (e \in E).$$

From this inequality and (6.24) follows that

$$\ell'(e) - \ell(e) \le \left\lfloor \frac{b_{\mathcal{F}}(e) - a_{\mathcal{F}}(e)}{nk} \right\rfloor \le \left\lfloor \frac{u(e)-\ell(e)}{n} \right\rfloor,$$

i.e., the former inequality in (6.23) holds.

6.4.4 Proof of Theorem 6.6 (ii)

We can compute a vector in $\mathcal{F}^\bullet$ by the following algorithm.

Let x_0 be a given vector in $\mathcal{F}$ and assume for simplicity that $E = \{1, 2, \dots, n\}$. For $i = 1, 2, \dots, n$, we iteratively define a vector $x_i \in \mathcal{F}$ as follows:

- if $a^\bullet_{\mathcal{F}}(i) \le x_{i-1}(i) \le b^\bullet_{\mathcal{F}}(i)$, then set $x_i = x_{i-1}$.
- if $x_{i-1}(i) < a^\bullet_{\mathcal{F}}(i)$, then let x_i be a vector in $\mathcal{F}$ which maximizes the value $x_i(i)$ under the constraints $x_i(i) \le b^\bullet_{\mathcal{F}}(i)$ and $a^\bullet_{\mathcal{F}}(e) \le x_i(e) \le b^\bullet_{\mathcal{F}}(e)$ $(e = 1, 2, \dots, i-1)$.
- if $x_{i-1}(i) > b^\bullet_{\mathcal{F}}(i)$, then let x_i be a vector in $\mathcal{F}$ which minimizes the value $x_i(i)$ under the constraints $x_i(i) \ge a^\bullet_{\mathcal{F}}(i)$ and $a^\bullet_{\mathcal{F}}(e) \le x_i(e) \le b^\bullet_{\mathcal{F}}(e)$ $(e = 1, 2, \dots, i-1)$.

By the statement (i) of Theorem 6.6, we see that the set

$$\mathcal{F}_i \equiv \mathcal{F} \cap \{x \mid a^\bullet_{\mathcal{F}}(e) \le x_i(e) \le b^\bullet_{\mathcal{F}}(e) \ (e = 1, 2 \dots, i)\}$$

is nonempty for all $i = 1, 2, \dots, n$. Therefore, the vector x_i is contained in $\mathcal{F}_i$; in particular, we have $x_n \in \mathcal{F}_n = \mathcal{F}^\bullet$.

By Theorem 5.2, each iteration of the algorithm can be done in $\mathrm{O}(nk \log \Phi(\mathcal{F}))$ time since $\mathcal{F}_i$ is a finite N_k-neighbor system. Hence, a vector in $\mathcal{F}^\bullet$ can be found in $\mathrm{O}(n^2 k \log \Phi(\mathcal{F}))$ time.

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