Noname manuscript No.
(will be inserted by the editor)

Matroid rank functions and discrete concavity

Akiyoshi Shioura

Received: date / Accepted: date

Abstract We discuss the relationship between matroid rank functions and a concept of discrete concavity called M$^\natural$-concavity. It is known that a matroid rank function and its weighted version called a weighted rank function are M$^\natural$-concave functions, while the (weighted) sum of matroid rank functions is not M$^\natural$-concave in general. We present a sufficient condition for a weighted sum of matroid rank functions to be an M$^\natural$-concave function, and show that every weighted rank function can be represented as a weighted sum of matroid rank functions satisfying this condition.

1 Introduction

The concept of matroid is a combinatorial structure which enjoys various nice properties, and it is deeply related with well-solvability of combinatorial optimization problems. A *matroid* $M = (E, \mathcal{F})$ is defined as a pair of a finite set E and a set family $\mathcal{F} \subseteq 2^E$ satisfying the following conditions:

(I0) $\emptyset \in \mathcal{F}$,
(I1) $I \subseteq J \in \mathcal{F}$ implies $I \in \mathcal{F}$,
(I2) $\forall I, J \in \mathcal{F},\ |I| < |J|,\ \exists u \in J \setminus I : I \cup \{u\} \in \mathcal{F}$.

The set E is called a *ground set* and each $I \in \mathcal{F}$ is called an *independent set*. In addition to this definition by independent sets, matroids can also be defined in several different ways by using bases, circuits, rank functions, etc.

This work is partially supported by a Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

Akiyoshi Shioura
Graduate School of Information Sciences,
Tohoku University, Sendai 980-8579, Japan
E-mail: shioura@dais.is.tohoku.ac.jp

Given a matroid $M = (E, \mathcal{F})$, its *rank function* is a function $\rho : 2^E \to \mathbb{Z}_+$ defined by

$$\rho(X) = \max\{|Y| \mid Y \in \mathcal{F},\ Y \subseteq X\} \quad (X \subseteq E). \tag{1.1}$$

Rank function ρ satisfies the following properties:

(R1) $\forall X \subseteq E : 0 \leq \rho(X) \leq |X|$,
(R2) ρ is monotone nondecreasing, i.e., $X \subseteq Y$ implies $\rho(X) \leq \rho(Y)$,
(R3) ρ is submodular, i.e., $\forall X, Y \subseteq E : \rho(X) + \rho(Y) \geq \rho(X \cap Y) + \rho(X \cup Y)$.

Moreover, these conditions characterize rank functions of matroids (see, e.g., [16, 18, 21]). In this paper, we investigate matroid rank functions from the viewpoint of discrete convex analysis.

Discrete convex analysis is a theoretical framework for well-solved combinatorial optimization problems introduced by Murota (see [12]; see also [13]), where the concepts of discrete convexity/concavity called M$^\natural$-convexity/M$^\natural$-concavity play central roles. The concepts of M$^\natural$-convexity/M$^\natural$-concavity are variants of M-convexity/M-concavity, originally introduced for functions defined over the integer lattice points by Murota and Shioura [15]. In this paper, we mainly consider M$^\natural$-concavity for set functions.

A set function $f : 2^E \to \mathbb{R}$ is said to be *M$^\natural$-concave* if it satisfies the following condition:

(M$^\natural$-EXC) for every $X, Y \subseteq E$ and every $u \in X \setminus Y$, either (i) or (ii) (or both) holds:

(i) $f(X) + f(Y) \leq f(X - u) + f(Y + u)$,
(ii) $\exists v \in Y \setminus X : f(X) + f(Y) \leq f(X - u + v) + f(Y + u - v)$,

where $X - u + v$ (resp., $Y + u - v$) is a short-hand notation for $(X \setminus \{u\}) \cup \{v\}$ (resp., $(Y \cup \{u\}) \setminus \{v\}$). It is shown that M$^\natural$-concavity for set functions is equivalent to the gross substitutes property in mathematical economics [9], and that M$^\natural$-concave functions constitute a proper subclass of submodular functions (see [12]). M$^\natural$-concavity for set functions is closely related to the concept of valuated matroid by Dress and Wenzel [3]; an M$^\natural$-concave function is defined over subsets of a finite set, while a valuated matroid is a function defined over bases of a matroid. It is noted that the sum of an M$^\natural$-concave function and a linear function is again an M$^\natural$-concave function, while the sum of two (or more) M$^\natural$-concave functions is not M$^\natural$-concave in general.

In this paper, we discuss the relationship between matroid rank functions and M$^\natural$-concavity. It is known that every matroid rank function is M$^\natural$-concave [8]. Moreover, a weighted version of matroid rank function called a *weighted rank function* is also M$^\natural$-concave [19], where a weighted rank function means a function $\rho_w : 2^E \to \mathbb{R}_+$ expressed as

$$\rho_w(X) = \max\{w(Y) \mid Y \in \mathcal{F},\ Y \subseteq X\} \qquad (X \subseteq E) \tag{1.2}$$

with a matroid $M = (E, \mathcal{F})$ and a nonnegative vector $w \in \mathbb{R}^E_+$. Here, we use the notation $w(Y) = \sum_{v \in Y} w(v)$. Note that a rank function in (1.1) is the weighted rank function with $w = (1, 1, \ldots, 1)$.

We also consider the weighted sum of matroid rank functions, which is called a *matroid-rank-sum function* in [2,5,6]. That is, a set function $f : 2^E \to \mathbb{R}_+$ is a matroid-rank-sum function if f can be represented as

$$f(X) = \sum_{i=1}^{k} \alpha_i \rho_i(X) \tag{1.3}$$

by using positive integer k, matroid rank functions $\rho_i : 2^E \to \mathbb{Z}_+$ ($i = 1, 2, \ldots, k$), and nonnegative real numbers $\alpha_i \in \mathbb{R}_+$ ($i = 1, 2, \ldots, k$).

Although a matroid-rank-sum function (1.3) is not M$^\natural$-concave in general, we derive a sufficient condition for a matroid-rank-sum function to be M$^\natural$-concave. For two matroids M_i ($i = 1, 2$) with rank functions $\rho_i : 2^E \to \mathbb{Z}_+$, we say that matroid M_1 is a *strong quotient* of M_2 if the rank functions satisfy the following condition:

$$\rho_1(X) - \rho_1(Y) \leq \rho_2(X) - \rho_2(Y) \qquad (\forall Y \subseteq \forall X \subseteq E); \tag{1.4}$$

in this case, we also say that the rank function ρ_1 is a strong quotient of ρ_2. We show that a matroid-rank-sum function in (1.3) is M$^\natural$-concave if the following condition holds (see Theorem 3):

(SQ) ρ_i is a strong quotient of ρ_{i+1} for each $i = 1, 2, \ldots, k-1$.

In addition, we show that every weighted rank function (1.2) can be represented as a weighted sum of matroid rank functions that satisfy the condition (SQ) (see Theorem 4). Hence, the results obtained in this paper are summarized as follows:

the set of weighted rank functions (1.2)
$\subseteq$ the set of matroid-rank-sum functions (1.3) with (SQ)
$\subseteq$ the set of M$^\natural$-concave functions.

This research is motivated by the submodular welfare maximization problem in combinatorial auctions, where matroid-rank-sum functions are regarded as a class of submodular functions with some useful properties (see [1,2,5,6]); indeed, they contain as special cases many concrete examples of submodular functions in this context. The submodular welfare maximization problem is NP-hard in general, even if the objective function is a matroid-rank-sum function. On the other hand, the problem can be solved exactly in polynomial time if the objective function is M$^\natural$-concave (see, e.g., [11]). Hence, the results in this paper shows that matroid-rank-sum functions with (SQ) constitute a tractable class of objective functions in the submodular welfare maximization problem.

2 Preliminaries

In this section we review some properties of matroids and M$^\natural$-concave functions, which will be used in the proofs of Section 3.

2.1 Matroids

A matroid $M = (E, \mathcal{F})$ is given as a pair of a ground set E and a family $\mathcal{F} \subseteq 2^E$ of independent sets. A family of independent sets of a matroid can be characterized by the following exchange property (see, e.g., [15, Remark 5.2]):

(G-EXC) $\forall I, J \in \mathcal{F}$, $\forall u \in I \setminus J$, (i) or (ii) (or both) holds:

$$\text{(i) } I - u, J + u \in \mathcal{F}, \quad \text{(ii) } \exists v \in J \setminus I : I - u + v, J + u - v \in \mathcal{F}.$$

Proposition 1 *A nonempty set family $\mathcal{F} \subseteq 2^E$ is the family of independent sets of a matroid if and only if $\emptyset \in \mathcal{F}$ and $\mathcal{F}$ satisfies* (G-EXC)*.*

More generally, the property (G-EXC) defines the concept of *generalized matroid* [20] (see also [7]); that is, a nonempty set family $\mathcal{F} \subseteq 2^E$ is called a generalized matroid if it satisfies (G-EXC) (see [15, Remark 5.2]). Proposition 1 shows that a family of independent sets of a matroid is equivalent to a generalized matroid containing the empty set.

The rank function $\rho : 2^E \to \mathbb{Z}_+$ of a matroid $M = (E, \mathcal{F})$ given by (1.1) satisfies the conditions (R1), (R2), and (R3), as mentioned in Introduction. We will also use the following property of rank functions (see, e.g., [16, Lemma 1.4.3]).

Proposition 2 *Let $\rho : 2^E \to \mathbb{Z}_+$ be a rank function of a matroid. Then, $\rho(X + u) - \rho(X) \in \{0, 1\}$ holds for every $X \subseteq E$ and $u \in E \setminus X$.*

In a linear optimization on a matroid, the optimal value can be expressed by using a matroid rank function (see, e.g., [18, Theorem 40.2]).

Proposition 3 *Let $M = (E, \mathcal{F})$ be a matroid and $w \in \mathbb{R}^E_+$ be a nonnegative vector. Suppose that $E = \{e_1, e_2, \ldots, e_n\}$ with $n = |E|$ and $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_n) \geq 0$. Then, it holds that*

$$\max\{w(X) \mid X \in \mathcal{F}\} = \sum_{i=1}^{n} (w(e_i) - w(e_{i+1}))\rho(E_i),$$

where $E_i = \{e_1, \ldots, e_i\}$ $(i = 1, 2, \ldots, n)$ and $w(e_{n+1}) = 0$.

For two matroids M_i $(i = 1, 2)$ with rank functions $\rho_i : 2^E \to \mathbb{Z}_+$, we say that M_1 is a *strong quotient* of M_2 if rank functions ρ_1 and ρ_2 satisfy the condition (1.4); in this case, we also say that the rank function ρ_1 is a strong quotient of ρ_2. A pair of matroids with strong-quotient relation can be obtained from a single matroid by deletion and contraction. Let $\widetilde{M} = (\widetilde{E}, \widetilde{\mathcal{F}})$ be a matroid with $E \subseteq \widetilde{E}$, and suppose that $X = \widetilde{E} \setminus E$ is an independent set of $\widetilde{M}$. We define set families $\widetilde{\mathcal{F}} \setminus X, \widetilde{\mathcal{F}}/X \subseteq 2^E$ by

$$\widetilde{\mathcal{F}} \setminus X = \{Y \setminus X \mid Y \in \widetilde{\mathcal{F}}\}, \qquad \widetilde{\mathcal{F}}/X = \{Y \setminus X \mid Y \in \widetilde{\mathcal{F}},\ X \subseteq Y\}.$$

Then, both of $(E, \widetilde{\mathcal{F}} \setminus X)$ and $(E, \widetilde{\mathcal{F}}/X)$ are matroids on the ground set E. We say that $(E, \widetilde{\mathcal{F}} \setminus X)$ and $(E, \widetilde{\mathcal{F}}/X)$ are matroids obtained from $\widetilde{M}$ by *deleting* X and *contracting* X, respectively.

Proposition 4 ([21, 22]) *$(E, \widetilde{\mathcal{F}}/X)$ is a strong quotient of $(E, \widetilde{\mathcal{F}} \setminus X)$, and every strong-quotient pair of matroids can be obtained in this way.*

2.2 M$^\natural$-concave functions

A set function $f : 2^E \to \mathbb{R}$ is said to be *M$^\natural$-concave* if it satisfies the condition (M$^\natural$-EXC). The concept of M$^\natural$-concavity is originally introduced for functions defined on integer lattice points (see, e.g., [12]), and the present definition of M$^\natural$-concavity for set functions can be obtained by specializing the original definition through the one-to-one correspondence between set functions and functions defined on 0-1 vectors.

It is known that every M$^\natural$-concave function is a submodular function (cf. [12]). Moreover, an M$^\natural$-concave function can be regarded a submodular function with an additional combinatorial property.

Theorem 1 (cf. [9, 17]) *A function $f : 2^E \to \mathbb{R}$ is M$^\natural$-concave if and only if it is a submodular function satisfying the following condition:*

$$
\begin{aligned}
&f(X \cup \{u, v\}) + f(X \cup \{t\}) \\
&\le \max\{f(X \cup \{u, t\}) + f(X \cup \{v\}), f(X \cup \{v, t\}) + f(X \cup \{u\})\} \\
&\qquad\qquad\qquad \textit{for every } X \subseteq E \textit{ and every distinct } u, v, t \in E \setminus X. \quad (2.1)
\end{aligned}
$$

Proof It is shown [9] that M$^\natural$-concavity for a set function f is equivalent to the gross-substitutes property (see, e.g., [9, 17] for the definition of the gross-substitutes property), while the gross-substitutes property for f can be characterized by the combination of submodularity and the condition (2.1), as shown in [17]. □

The condition (2.1) in Theorem 1 can be rewritten as follows.

Proposition 5 *For a function $f : 2^E \to \mathbb{R}$, the condition (2.1) holds if and only if for every $X \subseteq E$ and every distinct $u, v, t \in E \setminus X$, the maximum among the three values $f(X \cup \{u, v\}) + f(X \cup \{t\})$, $f(X \cup \{u, t\}) + f(X \cup \{v\})$, and $f(X \cup \{v, t\}) + f(X \cup \{u\})$ is attained by at least two of them.*

3 Relationship among weighted rank functions, matroid-rank-sum functions, and M$^\natural$-concave functions

We denote

$$
\begin{aligned}
\mathcal{M}^\natural &= \{f \mid f : 2^E \to \mathbb{R}_+ \text{ is M}^\natural\text{-concave}\},\\
\mathcal{R}_{\mathrm{wr}} &= \{f \mid f : 2^E \to \mathbb{R}_+ \text{ is a weighted rank function}\},\\
\mathcal{R}_{\text{mrs-sq}} &= \{f \mid f : 2^E \to \mathbb{R}_+ \text{ is a matroid-rank-sum function}\\
&\qquad\qquad\qquad\qquad \text{with the condition (SQ)}\}.
\end{aligned}
$$

We will prove that the following relations hold:

$$\mathcal{R}_{\mathrm{wr}} \subseteq \mathcal{R}_{\text{mrs-sq}} \subseteq \mathcal{M}^\natural.$$

3.1 M$^\natural$-concavity of weighted rank functions

We firstly review the known results that weighted rank functions as well as matroid rank functions are M$^\natural$-concave. This shows that $\mathcal{R}_{\rm wr} \subseteq \mathcal{M}^\natural$ holds.

Theorem 2 ([19, Theorem 1.2]) *For a matroid $M = (E, \mathcal{F})$ and a nonnegative vector $w \in \mathbb{R}^E_+$, the weighted rank function $\rho_w : 2^E \to \mathbb{R}_+$ given by* (1.2) *is an M$^\natural$-concave function.*

Proof For readers' convenience, we here give an elementary proof by Murota [14]. Take $X, Y \subseteq E$ and $u \in X \setminus Y$. Let $I, J \in \mathcal{F}$ be independent subsets of X and Y, respectively, such that $\rho_w(X) = w(I)$ and $\rho_w(Y) = w(J)$.

If $u \notin I$, then

$$\rho_w(X - u) \geq w(I) = \rho_w(X), \quad \rho_w(Y + u) \geq w(J) = \rho_w(Y),$$

which implies (i) in (M$^\natural$-EXC). So assume $u \in I$. If $J + u \in \mathcal{F}$, then

$$\rho_w(X-u) \geq w(I-u) = \rho_w(X)-w(u), \quad \rho_w(Y+u) \geq w(J+u) = \rho_w(Y)+w(u),$$

which implies (i) in (M$^\natural$-EXC). So assume $J + u \notin \mathcal{F}$. Then, by (G-EXC) for $\mathcal{F}$ (see Proposition 1), there exists $v \in J \setminus I$ such that $I - u + v, J + u - v \in \mathcal{F}$. If $v \in X$, then $I - u + v \subseteq X - u$, $J + u - v \subseteq Y + u$, and hence

$$\rho_w(X - u) \geq w(I - u + v) = \rho_w(X) - w(u) + w(v),$$
$$\rho_w(Y + u) \geq w(J + u - v) = \rho_w(Y) + w(u) - w(v),$$

which implies (i) in (M$^\natural$-EXC). If $v \notin X$, then $v \in Y \setminus X$, and

$$\rho_w(X - u + v) \geq w(I - u + v) = \rho_w(X) - w(u) + w(v),$$
$$\rho_w(Y + u - v) \geq w(J + u - v) = \rho_w(Y) + w(u) - w(v),$$

which implies (ii) in (M$^\natural$-EXC). □

By setting $w = (1, 1, \ldots, 1)$ in Theorem 2, we obtain the following property:

Corollary 1 ([8, p. 51]) *For a matroid $M = (E, \mathcal{F})$, its rank function $\rho : 2^E \to \mathbb{Z}_+$ given by* (1.1) *is an M$^\natural$-concave function.*

3.2 Matroid-rank-sum functions and M$^\natural$-concave functions

We now prove the inclusion $\mathcal{R}_{\text{mrs-sq}} \subseteq \mathcal{M}^\natural$.

Theorem 3 *Let k be a positive integer. For matroid rank functions $\rho_i : 2^E \to \mathbb{Z}_+$ $(i = 1, 2, \ldots, k)$ and nonnegative real numbers $\alpha_i \in \mathbb{R}_+$ $(i = 1, 2, \ldots, k)$, the matroid-rank-sum function $f : 2^E \to \mathbb{R}_+$ given by* (1.3) *is a monotone-nondecreasing M$^\natural$-concave function if the condition* (SQ) *holds.*

The following is the key property for the proof of Theorem 3. For submodular functions $f, g : 2^E \to \mathbb{R}$, we say, following [10], that f is a *strong quotient* of g if

$$f(X) - f(Y) \leq g(X) - g(Y) \qquad (\forall Y \subseteq \forall X \subseteq E).$$

Lemma 1 *Let $\rho : 2^E \to \mathbb{Z}_+$ be the rank function of a matroid, and $g : 2^E \to \mathbb{R}$ be a monotone-nondecreasing M$^\natural$-concave function. If g is a strong quotient of ρ, then the function $f : 2^E \to \mathbb{R}$ given by*

$$f(X) = \alpha g(X) + \beta \rho(X) \qquad (X \subseteq E)$$

with nonnegative real numbers $\alpha, \beta \in \mathbb{R}_+$ is a monotone-nondecreasing M$^\natural$-concave function.

Proof By Theorem 1 and Proposition 5, it suffices to show that f is monotone nondecreasing and submodular, and satisfies the condition that

(∗) the maximum in $\{\hat{f}(u), \hat{f}(v), \hat{f}(t)\}$ is attained by at least two of them

for every $X \subseteq E$ and every distinct $u, v, t \in E \setminus X$, where

$$\hat{f}(u) = f(X \cup \{v, t\}) + f(X \cup \{u\}),\ \hat{f}(v) = f(X \cup \{u, t\}) + f(X \cup \{v\}),$$
$$\hat{f}(t) = f(X \cup \{u, v\}) + f(X \cup \{t\}).$$

Recall that the rank function ρ is monotone nondecreasing, submodular, and M$^\natural$-concave by (R2), (R3), and Corollary 1. Since g is M$^\natural$-concave, it is submodular by Theorem 1. Hence, monotonicity and submodularity of f follow from those of g and ρ.

To prove the condition (∗), we fix $X \subseteq E$ and distinct elements $u, v, t \in E \setminus X$. We define $\hat{g}(u), \hat{g}(v), \hat{g}(t)$ and $\hat{\rho}(u), \hat{\rho}(v), \hat{\rho}(t)$ in a similar way as $\hat{f}(u), \hat{f}(v), \hat{f}(t)$. Note that $\hat{f}(s) = \alpha\hat{g}(s) + \beta\hat{\rho}(s)$ holds for $s \in \{u, v, t\}$.

Since g and ρ are M$^\natural$-concave functions, Theorem 1 and Proposition 5 imply that the maximum in $\{\hat{g}(u), \hat{g}(v), \hat{g}(t)\}$ is attained by at least two of them, and that the maximum in $\{\hat{\rho}(u), \hat{\rho}(v), \hat{\rho}(t)\}$ is attained by at least two of them. Hence, (∗) holds immediately if the following condition holds:

$$\exists s \in \{u, v, t\} : \min\{\hat{g}(u), \hat{g}(v), \hat{g}(t)\} = \hat{g}(s),\ \min\{\hat{\rho}(u), \hat{\rho}(v), \hat{\rho}(t)\} = \hat{\rho}(s). \tag{3.1}$$

In the following, we assume that the condition (3.1) does not hold and derive a contradiction.

We may assume, without loss of generality, that

$$\hat{g}(u) = \hat{g}(t) > \hat{g}(v), \qquad \hat{\rho}(t) = \hat{\rho}(v) > \hat{\rho}(u).$$

Since $\hat{g}(u) > \hat{g}(v)$, we have

$$g(X \cup \{v, t\}) - g(X \cup \{v\}) > g(X \cup \{u, t\}) - g(X \cup \{u\}). \tag{3.2}$$

Similarly, $\hat{\rho}(v) > \hat{\rho}(u)$ implies that

$$1 = \rho(X \cup \{u, t\}) - \rho(X \cup \{u\}) > \rho(X \cup \{v, t\}) - \rho(X \cup \{v\}) = 0, \tag{3.3}$$

where the two equalities are by Proposition 2. Since g is a strong quotient of ρ, it holds that

$$\begin{aligned}0 &= \rho(X \cup \{v,t\}) - \rho(X \cup \{v\}) \\ &\geq g(X \cup \{v,t\}) - g(X \cup \{v\}) > g(X \cup \{u,t\}) - g(X \cup \{u\}),\end{aligned}$$

where the equality is by (3.3) and the second inequality is by (3.2). Hence, we have $g(X \cup \{u,t\}) - g(X \cup \{u\}) < 0$, which contradicts the assumption that g is monotone nondecreasing. □

We give a proof of Theorem 3 by using Lemma 1.

Proof (of Theorem 3) We prove the claim by induction on the integer k. If $k = 1$ or $\alpha_1 = \alpha_2 = \cdots = \alpha_{k-1} = 0$, then the claim follows from the property (R2) and Corollary 1. Hence, we assume $k \geq 2$ and $\alpha = \sum_{i=1}^{k-1} \alpha_i > 0$. Define $g : 2^E \to \mathbb{R}$ by

$$g(X) = \frac{1}{\alpha} \sum_{i=1}^{k-1} \alpha_i \rho_i(X) \ (X \subseteq E).$$

Then, g is a monotone-nondecreasing $\mathrm{M}^\natural$-concave function by the induction hypothesis. The condition (SQ) implies that ρ_i is a strong quotient of ρ_k for $i = 1, 2, \ldots, k-1$, i.e.,

$$\rho_i(X) - \rho_i(Y) \leq \rho_k(X) - \rho_k(Y) \qquad (\forall Y \subseteq \forall X \subseteq E).$$

Hence, we have

$$\begin{aligned}g(X) - g(Y) &= \frac{1}{\alpha} \sum_{i=1}^{k-1} \alpha_i(\rho_i(X) - \rho_i(Y)) \\ &\leq \rho_k(X) - \rho_k(Y) \qquad (\forall Y \subseteq \forall X \subseteq E),\end{aligned}$$

i.e., g is a strong quotient of ρ_k. Since $f = \alpha g + \alpha_k \rho_k$, the function f is monotone nondecreasing and $\mathrm{M}^\natural$-concave by Lemma 1. □

Remark 1 It is noted that a matroid-rank-sum function without the condition (SQ) is not $\mathrm{M}^\natural$-concave in general, which can be shown as follows by using a well-known fact that the intersection of matroids is not a matroid.

For $i = 1, 2$, let $M_i = (E, \mathcal{F}_i)$ be a matroid with rank function $\rho_i : 2^E \to \mathbb{Z}$, and assume that $(E, \mathcal{F}_1 \cap \mathcal{F}_2)$ is not a matroid. We show that a function $f : 2^E \to \mathbb{R}$ given by $f = \rho_1 + \rho_2$ is not $\mathrm{M}^\natural$-concave, on the basis of the following fact (cf. [12]):

> for an $\mathrm{M}^\natural$-concave function $f : 2^E \to \mathbb{R}$ and a vector $p \in \mathbb{R}^E$, the set of maximizers $\arg\max\{f(X) - p(X) \mid X \subseteq E\}$ is a generalized matroid.

For $i = 1, 2$, we have $\rho_i(X) - |X| \leq 0$ for every $X \subseteq E$, and $\rho_i(X) - |X| = 0$ holds if and only if X is an independent set of M_i. Therefore, it holds that

$$\arg\max\{\rho_i(X) - |X| \mid X \subseteq E\} = \mathcal{F}_i.$$

For $p = (2, 2, \ldots, 2)$, we have

$$\begin{aligned} &\arg\max\{f(X) - p(X) \mid X \subseteq E\} \\ &= \arg\max\{\rho_1(X) + \rho_2(X) - 2|X| \mid X \subseteq E\} \\ &= \arg\max\{(\rho_1(X) - |X|) + (\rho_2(X) - |X|) \mid X \subseteq E\}. \end{aligned} \tag{3.4}$$

Putting

$$\mathcal{F} = \arg\max\{\rho_1(X) - |X| \mid X \subseteq E\} \cap \arg\max\{\rho_2(X) - |X| \mid X \subseteq E\},$$

we have

$$\mathcal{F} = \mathcal{F}_1 \cap \mathcal{F}_2, \tag{3.5}$$

which implies, in particular, that $\mathcal{F}$ is nonempty. Hence, we have

$$\arg\max\{(\rho_1(X) - |X|) + (\rho_2(X) - |X|) \mid X \subseteq E\} = \mathcal{F}. \tag{3.6}$$

From (3.4), (3.5), and (3.6) follows that

$$\arg\max\{f(X) - p(X) \mid X \subseteq E\} = \mathcal{F}_1 \cap \mathcal{F}_2,$$

which is not a family of independent sets of a matroid. Moreover, $\mathcal{F}_1 \cap \mathcal{F}_2$ is not a generalized matroid, which follows from Proposition 1 since $\emptyset \in \mathcal{F}_1 \cap \mathcal{F}_2$. Hence, the function f is not M$^\natural$-concave. □

3.3 Matroid-rank-sum functions and weighted rank functions

We finally show that $\mathcal{R}_{\text{wr}}$ is contained in $\mathcal{R}_{\text{mrs-sq}}$. We denote $n = |E|$.

Theorem 4 *Let $\rho_w : 2^E \to \mathbb{R}_+$ be a weighted rank function given by* (1.2) *with a matroid $M = (E, \mathcal{F})$ and a nonnegative vector $w \in \mathbb{R}^E_+$. Then, there exist matroid rank functions $\rho_i : 2^E \to \mathbb{Z}_+$ $(i = 1, 2, \ldots, n)$ satisfying the condition* (SQ) *and nonnegative real numbers $\alpha_i \in \mathbb{R}_+$ $(i = 1, 2, \ldots, n)$ such that*

$$\rho_w = \sum_{i=1}^{n} \alpha_i \rho_i.$$

To prove Theorem 4, we use the following property. To the end of this section, we assume, without loss of generality, that $E = \{e_1, e_2, \ldots, e_n\}$ and $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_n) \geq 0$, and denote $E_i = \{e_1, \ldots, e_i\}$ $(i = 1, 2, \ldots, n)$.

Lemma 2 *For every $X \subseteq E$, we have*

$$\rho_w(X) = \sum_{i=1}^{n} (w(e_i) - w(e_{i+1}))\rho(X \cap E_i). \tag{3.7}$$

Proof Let $M_X = (X, \mathcal{F}_X)$ be the matroid obtained from M by restriction to X, i.e., $\mathcal{F}_X$ is given by $\mathcal{F}_X = \{Y \cap X \mid Y \in \mathcal{F}\}$. Let $\rho_X : 2^X \to \mathbb{Z}_+$ be the rank function of M_X. Then, we have $\rho_X(Y) = \rho(Y)$ for every $Y \subseteq X$.

Suppose that $X = \{e_{i_1}, e_{i_2}, \ldots, e_{i_t}\}$ with $t = |X|$, where $i_1 < i_2 < \cdots < i_t$. Then, Proposition 3 implies

$$\begin{aligned} \rho_w(X) &= \sum_{j=1}^{t} (w(e_{i_j}) - w(e_{i_{j+1}}))\rho_X(\{e_{i_1}, e_{i_2}, \ldots, e_{i_j}\}) \\ &= \sum_{j=1}^{t} (w(e_{i_j}) - w(e_{i_{j+1}}))\rho(X \cap E_{i_j}), \end{aligned} \tag{3.8}$$

where $w(e_{i_{t+1}}) = 0$. It is not difficult to see that the right-hand side in (3.8) is equal to the right-hand side in (3.7). □

Proof (of Theorem 4) We firstly show that ρ_w can be represented as a weighted sum of matroid rank functions. Note that this part of the proof is essentially the same as the one in [4, Corollary 2.6].

For $i = 1, 2, \ldots, n$, we define $\mathcal{F}_i = \{X \cap E_i \mid X \in \mathcal{F}\}$. Then, $M_i = (E, \mathcal{F}_i)$ is a matroid, and denote by $\rho_i : 2^E \to \mathbb{Z}_+$ the rank function of M_i. We have $\rho_i(X) = \rho(X \cap E_i)$ for every $X \subseteq E$ and $i = 1, 2, \ldots, n$. Hence, Lemma 2 implies that

$$\begin{aligned} \rho_w(X) &= \sum_{i=1}^{n} (w(e_i) - w(e_{i+1}))\rho(X \cap E_i) \\ &= \sum_{i=1}^{n} (w(e_i) - w(e_{i+1}))\rho_i(X). \end{aligned}$$

This shows that ρ_w is represented as a weighted sum of matroid rank functions ρ_i.

To conclude the proof, we show that the condition (SQ) holds, i.e., $M_i = (E, \mathcal{F}_i)$ is a strong quotient of $M_{i+1} = (E, \mathcal{F}_{i+1})$ for $i = 1, 2, \ldots, n-1$. Let e_0 be an element not contained in E, and let $\widetilde{\mathcal{F}}_{i+1}$ be a family of subsets of $E \cup \{e_0\}$ given by

$$\widetilde{\mathcal{F}}_{i+1} = \mathcal{F}_{i+1} \cup \{(X \cup \{e_0\}) \setminus \{e_{i+1}\} \mid X \in \mathcal{F}_{i+1},\ e_{i+1} \in X\}.$$

Then, $\widetilde{M}_{i+1} = (E \cup \{e_0\}, \widetilde{\mathcal{F}}_{i+1})$ is a matroid. Note that in the matroid $\widetilde{M}_{i+1}$, element e_0 is parallel to e_{i+1}. We see from the definition of M_i (resp., M_{i+1}) that the matroid M_i (resp., M_{i+1}) can be obtained from $\widetilde{M}_{i+1}$ by contracting e_0 (resp., by deleting e_0). Hence, M_i is a strong quotient of M_{i+1} by Proposition 4. □

Remark 2 To show that $\mathcal{R}_{\rm wr}$ is properly contained in $\mathcal{R}_{\rm mrs\text{-}sq}$, we present an example of matroid-rank-sum function which satisfies (SQ) but is not a weighted rank function.

Let $E = \{a, b, c\}$ and consider two matroids $M_i = (E, \mathcal{F}_i)$ $(i = 1, 2)$ on E, where

$$\mathcal{F}_1 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}, \qquad \mathcal{F}_2 = 2^E.$$

It can be shown that M_1 is a strong quotient of M_2. For $i = 1, 2$, let $\rho_i : 2^E \to \mathbb{Z}_+$ be the rank function of matroid M_i, and define $f : 2^E \to \mathbb{Z}$ by $f = \rho_1 + \rho_2$. Then, f is a matroid-rank-sum function with (SQ). Note that

$$f(X) = 2 \quad \text{if } |X| = 1, \qquad f(\{b, c\}) = 3.$$

Suppose, to the contrary, that f is a weighted rank function. Then, there exist a matroid $M = (E, \mathcal{F})$ and a weight vector $w \in \mathbb{R}^E_+$ such that

$$f(X) = \max\{w(Y) \mid Y \in \mathcal{F},\ Y \subseteq X\}.$$

Since $f(\{a\}) = f(\{b\}) = f(\{c\}) = 2$, we have $w(a) = w(b) = w(c) = 2$. This implies that the value of f should be a multiple of 2, while $f(\{b, c\}) = 3$, a contradiction. Hence, f is not a weighted rank function. □

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