Journal of the Operations Research
Society of Japan
20??, Vol. ??, No. ?, ???-???

Efficient Strategy Proof Fair Allocation Algorithms

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(Received November 16, 2004; Revised September 7, 2005)

Abstract We study a fair division problem with indivisible objects like jobs, houses, and one divisible good like money. Each individual is to be assigned with one object and a certain amount of money. The preferences of individuals over the objects are private information but individuals are assumed to have quasi-linear utilities in money. It is shown that there exist efficient algorithms for eliciting honest preferences and assigning the objects with money to individuals efficiently and fairly.

Keywords: Economics, indivisibility, fairness, efficiency, polynomial-time algorithms, strategy proof

1. Introduction

There is by now a large literature on fair allocation (or division) problems (in particular on divisible goods); see, e.g., Moulin [10], Young [21] and Fujishige [4, Chapter 5] and references therein. The existence problem of fair allocation with indivisible objects and money has been studied by Svensson [17], Maskin [9], Alkan et al. [1], Tadenuma and Thomson [19], Su [14], Yang [20], Sun and Yang [15] in quite general settings. Meanwhile, several efficient procedures have been proposed by Aragones [2], Klijn [7], and Haake et al. [6] to find fair allocations in the quasi-linear utility case. Although the strategic issue was explicitly raised in Alkan et al. [1, pp.1038–1039], none of these existing procedures can guarantee the elicitation of honest preferences from the individuals.

This paper addresses a fair division problem with indivisible objects like jobs, tasks, cars, houses, and one divisible good like money. The objects are to be distributed with a certain amount of money among a group of individuals in a way which is efficient and fair. The objects may be desirable like houses or undesirable like tasks which must be performed, and moreover the amount of money may be negative as, for example, costs to be shared by the individuals. The preferences of individuals over the objects are private information but individuals are assumed to have quasi-linear utilities in money. It is required that each individual be assigned with exactly one object even if it may be still unprofitable to him after a compensation is made. Each object j is associated with a maximum compensation limit $c(j)$. This upper bound $c(j)$ is known to all individuals prior to the announcement of their preferences over the objects and is independent of whatever preferences might be reported by the individuals. Unlike market economies in which the influence of any single individual is almost negligible compared with the market size, strategic issues arising from the current economic model are quite severe because the number of individuals is usually relatively small and therefore it is not inconceivable that some agents may have incentive to manipulate when they have advantage in doing so. In addition, since the preferences of each individual over the objects is private information and thus known only to himself, he is free to report anything at his discretion. He will reveal his true preferences only when doing so is in his interest.

In this paper we show that the fair allocation problem we deal with is essentially equivalent to the assignment problem plus the single-source shortest-path problem and then show that any algorithms for the latter two problems can be used to induce all the individuals to reveal truthfully their preferences over the objects, and to assign the objects with money to individuals efficiently and fairly. Those algorithms can be seen as a realization of the strategy proof fair allocation mechanism introduced by Sun and Yang [16] for more general environments. This mechanism is further studied in Svensson [18] and is closely related to the mechanisms given by Clark [3], Groves [5], and Leonard [8] in different contexts.

2. The Main Results

Let I_k denote the set of the first k positive integers and $\mathbb{R}$ the set of real numbers. Furthermore, let $\mathbb{R}^n$ stand for the n-dimensional Euclidean space.

Now we will briefly describe the fair allocation model which consists of n agents, n indivisible objects and a certain amount of money. The set of agents is denoted by I_n and the set of objects by N, where $N = I_n$. Objects could be any inherently indivisible goods like houses, cars, and could be positions or tasks as well. Each object j has an upper bound compensation limit $c(j)$ units of money. It is required that each agent should be assigned with exactly one object. All agents are assumed to have quasi-linear utilities in money. Each agent i places a monetary value, namely, a reservation value $V(i, j)$ on each object j. These values $V(i, j)$ are private information and thus known only to agent i himself. Thus, the set of all possible reservation value functions of each agent can be represented by the n-dimensional real space $\mathbb{R}^n$. A list $V = (V(1, \cdot), \cdots, V(n, \cdot))$ of all individual reservation value functions is called *a preference profile* or simply *a profile*. Let $\mathcal{V}$ denote the family of all profiles. A profile $V = (V(1, \cdot), \cdots, V(n, \cdot))$ is also written as $V = (V(i, \cdot), V(-i, \cdot))$ for $i \in I_n$, where $V(-i, \cdot)$ denotes $(V(1, \cdot), \cdots, V(n, \cdot))$ without $V(i, \cdot)$.

Let $C = (c(1), \cdots, c(n))$ be the vector of maximum compensations. This model is denoted by $E = [C, V(i, j), i \in I_n, j \in N]$ for a given profile $V \in \mathcal{V}$. An *allocation* (π, x) consists of a permutation π of the n objects and a function $x : N \to \mathbb{R}$ called a compensation scheme. The set of all allocations is denoted by $\mathcal{A}$. At the allocation (π, x), agent i gets object $\pi(i)$ and $x(\pi(i))$ units of money. In case $x(\pi(i))$ is negative, agent i has to pay the amount of $|x(\pi(i))|$. A compensation scheme $x : N \to \mathbb{R}$ is *feasible* if $x(j) \le c(j)$ for all $j \in N$. An allocation (π, x) is *fair* if

$$V(i, \pi(i)) + x(\pi(i)) \ge V(i, \pi(j)) + x(\pi(j)) \qquad (\forall i \in I_n,\ \forall j \in I_n).$$

A fair allocation (π, x) is *compatible* with the vector C if x is a feasible compensation scheme. A fair allocation is *Pareto optimal* if there exists no other allocation (ρ, y) such that $V(i, \rho(i)) + y(\rho(i)) \ge V(i, \pi(i)) + x(\pi(i))$ for every $i \in I_n$, and $V(j, \rho(j)) + y(\rho(j)) > V(j, \pi(j)) + x(\pi(j))$ for some $j \in I_n$ and $\sum_{j=1}^n x(j) = \sum_{j=1}^n y(j)$. It is easy to see that every fair allocation is Pareto optimal. For a given profile $V \in \mathcal{V}$, the set of compatible fair allocations is denoted by $CF(V)$. Clearly, $CF(V)$ is a subset of $\mathcal{A}$. For a given profile $V \in \mathcal{V}$, a compatible fair allocation (π, x) is *optimal* if for every compatible fair allocation (ρ, y) in $CF(V)$ it holds that $x \ge y$. The problem of finding an optimal fair allocation is called *the optimal fair allocation problem.*

A *mechanism* is a rule or a recipe that specifies an allocation for each profile $V \in \mathcal{V}$. In other words, we can regard a mechanism as a function $f : \mathcal{V} \to \mathcal{A}$; i.e., for every $V \in \mathcal{V}$, the mechanism f specifies an allocation $f(V) = (\pi, x)$. However, not every mechanism is economically sensible. In many situations, since the rule of a mechanism is written precisely and known to everyone, it is not inconceivable that some agents may have incentive to manipulate and abuse when they have advantage in doing so. Therefore, it is very important to design mechanisms that can induce people to behave honestly. In the current model, because the reservation values of each agent over the objects are known only to himself, he will report his true reservation values only when truth-telling advances his own economic interest. This implies that any viable mechanism must be strategy proof.

Precisely, a mechanism is *strategy proof* if no agent can make himself strictly better off by misreporting his reservation values over the objects, given that all other agents reveal their true reservation values. Formally, a mechanism $f : \mathcal{V} \to \mathcal{A}$ is strategy proof if, for every agent $i \in I_n$, every profile $V \in \mathcal{V}$, and every reservation value function $V'(i, \cdot) \in \mathbb{R}^n$ of agent i, f satisfies the following no-incentive-to misrepresent condition:

$$V(i, \pi(i)) + x(\pi(i)) \geq V(i, \rho(i)) + y(\rho(i)),$$

where $(\pi, x) = f(V)$ and $(\rho, y) = f((V'(i, \cdot), V(-i, \cdot))$. That is, in the face of a strategy proof mechanism, honesty is a best strategy for every individual.

A mechanism that always selects an optimal fair allocation for each profile $V \in \mathcal{V}$ is called *the optimal fair allocation mechanism.* As shown by Sun and Yang [16] in more general cases, this type of mechanism is of considerable interest in the sense that it is Pareto optimal, fair, and strategy proof. Svensson [18] has further studied this mechanism.

In the following we will show that the optimal fair allocation problem can be transformed to the assignment problem plus the single-source shortest-path problem, and then indicate that any algorithms for the latter two problems can be used to find an optimal fair allocation and thus can be seen as a realization of the optimal fair allocation mechanism.

Given a model $E = [C, V(i, j), i \in I_n, j \in N]$, we can formulate the corresponding maximum weight matching problem (P):

$$\begin{aligned}
\text{(P)} \quad \text{Maximize} \quad & \sum_{i \in I_n} \sum_{j \in N} V(i, j) z(i, j) \\
\text{subject to} \quad & \sum_{j \in N} z(i, j) = 1 \quad (\forall i \in I_n), \\
& \sum_{i \in I_n} z(i, j) = 1 \quad (\forall j \in N), \\
& z(i, j) \geq 0 \quad (\forall i \in I_n, \ \forall j \in N).
\end{aligned}$$

An integer optimal solution of this problem is called *an optimal assignment* and can be represented by a permutation π of the objects in N. That is, $z(i, \pi(i)) = 1$ for every $i \in I_n$ and $z(i, j) = 0$ otherwise. Such an optimal solution always exists and gives a maximum weight matching between agents and objects. Let Π denote the set of all optimal assignments of (P). The dual problem (D) of the above problem can be written as

$$\begin{aligned}
\text{(D)} \quad \text{Minimize} \quad & \sum_{i \in I_n} u(i) + \sum_{j \in N} v(j) \\
\text{subject to} \quad & u(i) + v(j) \geq V(i, j) \quad (\forall i \in I_n, \ \forall j \in N).
\end{aligned}$$

The set of optimal solutions (u, v) to the above problem is denoted by $OD(V)$. By the complementary slackness theorem in linear programming (see, e.g., Schrijver [13]), we have the following characterization of optimal solutions of (P) and (D).

Theorem 2.1. *Let π be a permutation representing a feasible assignment of* (P)*, and (u, v) be a feasible solution of* (D)*. Then, π and (u, v) are both optimal if and only if they satisfy the complementary slackness condition*

$$u(i) + v(\pi(i)) = V(i, \pi(i)),\ \forall\, i \in I_n. \tag{2.1}$$

Based upon this result, one can easily show an intimate relationship between fair allocations and the pairs of optimal solutions of (P) and (D) as follows.

Theorem 2.2.
(i) *Assume that (π, x) is a fair allocation. Then, π is an optimal assignment of* (P) *and the pair of vectors $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$ given by*

$$v = -x, \qquad u(i) = V(i, \pi(i)) + x(\pi(i)) \quad (i \in I_n)$$

is an optimal solution of (D)*.*
(ii) *Assume that π is an optimal assignment of* (P) *and (u, v) is an optimal solution of* (D)*. Then $(\pi, -v)$ is a fair allocation.*

Proof. (i) It is easy to see that π and (u, v) are feasible solutions of (P) and (D), respectively, and satisfy the complementary slackness condition (2.1). Hence, π and (u, v) are optimal solutions of (P) and (D), respectively.

(ii) From Theorem 2.1, the optimal assignment π of (P) and the optimal solution $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$ satisfy the following property:

$$\begin{array}{ll} u(i) + v(j) \geq V(i, j), & \forall i \in I_n,\ \forall j \in N, \\ u(i) + v(\pi(i)) = V(i, \pi(i)), & \forall i \in I_n. \end{array}$$

It follows immediately that

$$V(i, \pi(i)) - v(\pi(i)) \geq V(i, j) - v(j)$$

for every i and every j. Hence, the allocation $(\pi, -v)$ is a fair allocation. $\square$

Let

$$\begin{array}{rcl} \Psi & = & \{x \in \mathbb{R}^n \mid \ (u, -x) \text{ is an optimal solution of (D) for some } u \in \mathbb{R}^n\} \\ & = & \{x \in \mathbb{R}^n \mid \ (u, -x) \in OD(V) \text{ for some } u \in \mathbb{R}^n\}. \end{array}$$

As an immediate consequence of Theorems 2.1 and 2.2, we have the following characterization of compatible and optimal fair allocations.

Corollary 2.3.
(i) *The set of all fair allocations is given as*

$$\Pi \times \Psi = \{(\pi, x) \mid \pi \in \Pi,\ x \in \Psi\}.$$

(ii) *The set $CF(V)$ of all compatible fair allocations is given as*

$$CF(V) = \Pi \times \{x \mid x \in \Psi, x \leq C\}.$$

(iii) *(π, x) is an optimal fair allocation if and only if π is an assignment in Π and x is a maximal vector in the set $\{x \mid x \in \Psi, x \leq C\}$ with respect to the partial order $\leq$.*

Observe that the set $CM(V)$ of compatible compensation schemes

$$CM(V) = \{x \mid x \in \Psi, x \leq C\}$$

is independent of the choice of an optimal assignment π and as shown below can be described explicitly by a system of linear inequalities defined by agents' reservation values and any given optimal assignment. We see also that in order to find an optimal fair allocation, we only need to find an optimal assignment of (P) and a maximal vector in the set $CM(V)$. In the following we explain how to find such a maximal vector.

Let π be any given optimal assignment of (P), i.e., $\pi \in \Pi$. It follows from Theorem 2.1 that

$$\begin{aligned}
\Psi &= \{x \in \mathbb{R}^n \mid \ (u, -x) \in OD(V) \text{ for some } u \in \mathbb{R}^n\} \\
&= \{x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^n \text{ s.t. } u(i) - x(j) \geq V(i,j) \ (\forall i, j \in I_n), \\
&\qquad\qquad\qquad\qquad u(i) - x(\pi(i)) = V(i, \pi(i)) \ (\forall i \in I_n)\} \\
&= \{x \in \mathbb{R}^n \mid x(\pi(i)) - x(j) \geq -V(i, \pi(i)) + V(i,j) \ (\forall i, j \in I_n)\}.
\end{aligned}$$

Consequently, the set of compatible compensation schemes can be expressed as

$$\begin{aligned}
CM(V) &= \{x \in \mathbb{R}^n \mid x(\pi(i)) - x(j) \geq -V(i, \pi(i)) + V(i,j) \ (\forall i, j \in I_n), \\
&\qquad\qquad\qquad x(j) \leq c(j) \ (\forall j \in I_n)\}. \qquad (2.2)
\end{aligned}$$

The set $CM(V)$ can be further seen as an instance of the following polyhedra

$$\begin{aligned}
P &= \{x \in \mathbb{R}^n \mid l(i,j) \leq x(i) - x(j) \leq u(i,j) \ (\forall i, j \in I_n), \\
&\qquad\qquad a(j) \leq x(j) \leq b(j) \ (\forall j \in I_n)\}, \qquad (2.3)
\end{aligned}$$

where $l(i,j)$, $u(i,j)$, $a(j)$ and $b(j)$ are values contained in $\mathbb{R} \cup \{+\infty, -\infty\}$. For the set $CM(V)$ of the form (2.2), the parameters are given by $l(i,j) = -V(\pi^{-1}(i), i) + V(\pi^{-1}(i), j)$, and $u(i,j) = +\infty$ for $i, j \in I_n$, and $a(j) = -\infty$ and $b(j) = c(j)$ for $j \in I_n$.

The polyhedra P of the form (2.3) can be found in the dual of the single-source shortest-path problem and have been well studied in the literature (see, e.g., Murota [11] and Schrijver [13]). In particular, the set P is a lattice. That is, for any $x, y \in P$, we have $x \wedge y$, $x \vee y \in P$, where

$$\begin{aligned}
x \wedge y &= (\min\{x(1), y(1)\}, \cdots, \min\{x(n), y(n)\}), \\
x \vee y &= (\max\{x(1), y(1)\}, \cdots, \max\{x(n), y(n)\}).
\end{aligned}$$

Hence, the set P has the unique maximal vector with respect to the partial order $\leq$ if $b(j)$ is finite for all $j \in I_n$. It is known that the unique maximal vector can be found in polynomial time by solving a single-source shortest-path problem (see, e.g., Murota [11] and Murota and Tamura [12]).

The above discussion has shown that any algorithms for the assignment problem and the single-source shortest-path problem can be used to find an optimal fair allocation. Hence, for each profile $V \in \mathcal{V}$, such an algorithm will give an optimal fair allocation and can be therefore seen as a realization of the optimal fair allocation mechanism. In Theorem 3.1 of Sun and Yang [16], it is shown that any mechanism that can find an optimal fair allocation achieves simultaneously Pareto optimality, fairness, and strategy proofness. The rationale behind strategy proofness is that the optimal fair allocation gives every agent the maximal benefit he can possibly get under the compensation limits C and thus no one has incentive to misrepresent.

Acknowledgements

We are grateful to the anonymous referees for their helpful comments and suggestions.

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