A Note on the Equivalence Between Substitutability and $M^e$-convexity

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Abstract

The property of “substitutability” plays a key role in guaranteeing the existence of a stable solution in the stable marriage problem and its generalizations. On the other hand, the concept of $M^e$-convexity, introduced by Murota–Shioura (1999) for functions defined over the integer lattice, enjoys a number of nice properties that are expected of “discrete convexity” and provides with a natural model of utility functions. In this note, we show that $M^e$-convexity is characterized by two variants of substitutability.

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1 Introduction

Since the pioneering work on the stable marriage problem by Gale–Shapley [7], various generalizations and extensions of the stable marriage model have been proposed in the literature (see [1, 2, 3, 4, 6, 14, 15], etc.), where the property of “substitutability” for preferences plays a key role in guaranteeing the existence of a stable solution. On the other hand, the concept of M-convexity, introduced by Murota [8, 9] for functions defined over the integer lattice, enjoys a number of nice properties that are expected of “discrete convexity;” subsequently, its variant called M♮-convexity was introduced by Murota–Shioura [11]. Whereas M²-convex functions are conceptually equivalent to M-convex functions, the class of M♮-convex functions is strictly larger than that of M-convex functions. Furthermore, M♮-concave functions provide with a natural model of utility functions [10, 13, 16]. In particular, it is known that M♮-concavity is equivalent to the gross substitutes property, and that M♮-concavity implies submodularity. In this note, we discuss the close relationship between substitutability and M♮-convexity/M²-concavity.

Recently, Eguchi–Fujishige–Tamura [3] extended the stable marriage model to the framework with preferences represented by M²-concave utility functions, and showed the existence of a stable solution in their model (see also [2]). Their proof is based on the fact that M²-concave functions \( f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\} \) satisfy the following properties:

\[
\begin{align*}
\text{(SC}^1\text{)} & \quad \forall z_1, z_2 \in \mathbf{Z}^V \text{ with } z_1 \geq z_2 \text{ and } \arg\min \{f(x') \mid x' \leq z_2\} \neq \emptyset, \\
& \quad \forall x_1 \in \arg\min \{f(x') \mid x' \leq z_1\}, \exists x_2 \in \arg\min \{f(x') \mid x' \leq z_2\} \text{ such that } z_2 \land x_1 \leq x_2, \\
\text{(SC}^2\text{)} & \quad \forall z_1, z_2 \in \mathbf{Z}^V \text{ with } z_1 \geq z_2 \text{ and } \arg\min \{f(x') \mid x' \leq z_1\} \neq \emptyset, \\
& \quad \forall x_2 \in \arg\min \{f(x') \mid x' \leq z_2\}, \exists x_1 \in \arg\min \{f(x') \mid x' \leq z_1\} \text{ such that } z_2 \land x_1 \leq x_2,
\end{align*}
\]

where for \( x, y \in \mathbf{R}^V \) the vector \( x \land y \in \mathbf{R}^V \) is given by \( (x \land y)(w) = \min\{x(w), y(w)\} \) \( (w \in V) \). These properties can be regarded as substitutability for utility functions \( f \); indeed, (SC^1) and (SC^2) can be seen as generalizations of substitutability (persistence) in the sense of Alkan–Gale [1] for the choice function \( C(z) = \arg\min\{f(y) \mid y \leq z\} \).

Following the work by Eguchi–Fujishige–Tamura [3], Fujishige–Tamura [6] presented a common generalization of the stable marriage model and the assignment game model with M♮-concave utility functions. It is shown in [6] that the following properties of M♮-convex functions

\[
\begin{align*}
\text{(SC}^1\text{G}) & \quad \forall p \in \mathbf{R}^V, \ f[p] \text{ satisfies (SC}^1\text{),} \\
\text{(SC}^2\text{G}) & \quad \forall p \in \mathbf{R}^V, \ f[p] \text{ satisfies (SC}^2\text{),}
\end{align*}
\]

which can be seen as stronger versions of substitutability (SC^1) and (SC^2), play a key role in the proof of the existence of a stable solution in this model, where for \( p \in \mathbf{R}^V \) the function \( f[p] : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\} \) is defined by

\[
f[p](x) = f(x) + \sum_{w \in V} p(w)x(w) \quad (x \in \mathbf{Z}^V).
\]

The main aim of this note is to prove that each of (SC^1G) and (SC^2G) characterizes M♮-convexity of a function.

**Theorem 1.1.** Let \( f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\} \) be a function such that the effective domain \( \text{dom } f = \{x \in \mathbf{Z}^V \mid f(x) < +\infty\} \) is bounded. Then,

\[
f \text{ is M♮-convex } \iff (\text{SC}^1\text{G}) \iff (\text{SC}^2\text{G}).
\]
This theorem shows that $M^2$-concavity of utility functions is an essential assumption in the model of Fujishige–Tamura [6]. Combining Theorem 1.1 and the previous result [13, Theorem 11] clarifies the relationship between substitutability and the gross substitute property for utility functions. The equivalence in Theorem 1.1 was proven by Farooq–Tamura [5] in the special case where $\text{dom } f \subseteq \{0,1\}^V$, i.e., $f$ is a set function. In this note, we give a proof for a more general case where $\text{dom } f$ is bounded.

2 Preliminaries on $M^2$-convexity

In this section, we review the definition and fundamental properties of $M^2$-convex functions.

Throughout this paper, we assume that $V$ is a nonempty finite set. The sets of reals and integers are denoted by $\mathbb{R}$ and by $\mathbb{Z}$, respectively. For a vector $x = (x(w) \mid w \in V) \in Z^V$, we define

$$\text{supp}^+(x) = \{w \in V \mid x(w) > 0\}, \quad \text{supp}^-(x) = \{w \in V \mid x(w) < 0\}, \quad \text{supp}(x) = \{w \in V \mid x(w) \neq 0\},$$

where $\langle p, x \rangle = \sum_{w \in V} p(w)x(w)$, $\langle p, x \rangle = \sum_{w \in S} x(w)$ ($S \subseteq V$).

For any $u \in V$, the characteristic vector of $u$ is denoted by $\chi_u (\in \{0,1\}^V)$, i.e., $\chi_u (w) = 1$ if $w = u$ and $\chi_u (w) = 0$ otherwise. We also denote by $\chi_0$ the zero vector. For $x, y \in Z^V$ with $x \leq y$, we denote $[x, y]_Z = \{z \in Z^V \mid x \leq z \leq y\}$.

Let $f : Z^V \to \mathbb{R} \cup \{+\infty\}$ be a function. We denote the set of minimizers of $f$ by $\arg \min f = \{x \in Z^V \mid f(x) \leq f(y) (\forall y \in Z^V)\}$, which can be the empty set. For a vector $z \in Z^V$, we denote $X^*(f, z) = \arg \min \{f(x) \mid x \leq z\} (= \{x \in Z^V \mid x \leq z, f(x) \leq f(y) (\forall y \in Z^V \text{ with } y \leq z)\})$.

We call a function $f : Z^V \to \mathbb{R} \cup \{+\infty\}$ $M^2$-convex if it satisfies $\text{dom } f \neq \emptyset$ and (M$^2$-EXC):

$$\text{(M}^2\text{-EXC)} \forall x, y \in \text{dom } f, \forall u \in \text{supp}^+(x-y), \exists v \in \text{supp}^-(x-y) \cup \{0\}:$$

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v).$$


We also define the set version of $M^2$-convexity. A nonempty set $B \subseteq Z^V$ is said to be $M^2$-convex if its indicator function $\delta_B : Z^V \to \{0, +\infty\}$ defined by

$$\delta_B(x) = \begin{cases} 0 & \text{if } x \in B, \\ +\infty & \text{otherwise} \end{cases}$$

is $M^2$-convex. Equivalently, an $M^2$-convex set is defined as a nonempty set satisfying the exchange property (B$^2$-EXC$_\pm$):

$$\text{(B}^2\text{-EXC}_\pm) \forall x, y \in B, \forall u \in \text{supp}^+(x-y), \exists v, w \in \text{supp}^-(x-y) \cup \{0\} \text{ such that } x - \chi_u + \chi_v \in B$$

and $y + \chi_u - \chi_w \in B$.

**Theorem 2.1** ([11, 17]). A nonempty set $B \subseteq Z^V$ is $M^2$-convex if and only if it satisfies (B$^2$-EXC$_\pm$).

An $M^2$-convex function with bounded effective domain can be characterized by the sets of minimizers.

**Theorem 2.2** ([10, Theorem 6.30]). Let $f : Z^V \to \mathbb{R} \cup \{+\infty\}$ be a function such that $\text{dom } f$ is bounded. Then, $f$ is $M^2$-convex if and only if for each $p \in \mathbb{R}^V$ the set $\arg \min f[p]$ is $M^2$-convex.
3 Proofs

The implications “f is M-convex =⇒ (SC\_1\_G)” and “f is M-convex =⇒ (SC\_2\_G)” are shown in [3, 5, 6] (see also Section 4).

**Theorem 3.1.** An M\-convex function f : Z\^V → R ∪ {+∞} satisfies (SC\_1\_G) and (SC\_2\_G).

In this section, we prove the implications “(SC\_2\_G) =⇒ (SC\_1\_G)” and “(SC\_1\_G) =⇒ f is M\-convex.”

**Theorem 3.2.** Let f : Z\^V → R ∪ {+∞}.

(i) If f satisfies (SC\_2\_G), then f also satisfies (SC\_1\_G).

(ii) Suppose that dom f is bounded. If f satisfies (SC\_1\_G), then f is M\-convex.

Combining Theorems 3.1 and 3.2 yields Theorem 1.1, our main result.

### 3.1 Proof of “(SC\_2\_G) =⇒ (SC\_1\_G)”

We prove Theorem 3.2 (i).

Suppose that f satisfies (SC\_2\_G). Let p ∈ R\^V, z\_1, z\_2 ∈ Z\^V be any vectors satisfying z\_1 ≥ z\_2 and X\^*(f[p], z\_2) ≠ ∅, and x\_1\_\^∗ ∈ X\^*(f[p], z\_1). Also, let x\_2\_\^∗ ∈ X\^*(f[p], z\_2) be a vector minimizing the cardinality of the set supp\+(x\_2\_\^∗ - x\_1\_\^∗), and put S\+ = supp\+(x\_2\_\^∗ - x\_1\_\^∗). We assume that x\_2\_\^∗ maximizes the value x\_2\_∥(V \setminus S\+) among all vectors y ∈ X\^*(f[p], z\_2) with supp\+(y - x\_1\_\^∗) = S\+. We show that x\_2\_\^∗ satisfies the inequality z\_2 \wedge x\_1\_\^∗ ≤ x\_2\_\^∗.

For w ∈ S\+, we have min\{z\_2(w), x\_1\_\^∗(w)\} = x\_1\_\^∗(w) < x\_2\_\^∗(w) since x\_1\_\^∗(w) < x\_2\_\^∗(w) ≤ z\_2(w). Hence, it suffices to prove that

\[
\min\{z\_2(w), x\_1\_\^∗(w)\} ≤ x\_2\_\^∗(w) \quad (w ∈ V \setminus S\+). \tag{3.1}
\]

To show this, we define \(\tilde{z}_1, \tilde{z}_2 ∈ Z^V\) by

\[\tilde{z}_1 = x\_1\_\^∗ \lor x\_2\_\^∗, \quad \tilde{z}_2 = (x\_1\_\^∗ \lor x\_2\_\^∗) \wedge z\_2.\]

For \(i = 1, 2, x\_i\_\^∗ ∈ X\^*(f[p], \tilde{z}_i) ⊆ X\^*(f[p], z\_i)\) holds since \(x\_i\_\^∗ ≤ \tilde{z}_i ≤ z\_i\). As shown below, there exists a vector \(q ∈ R^V\) satisfying the following conditions:

\[
X\^*(f[q], \tilde{z}_1) ≠ ∅, \quad \text{and} \quad x(w) = x\_1\_\^∗(w) (w ∈ V \setminus S\+) \quad \text{for all} \quad x ∈ X\^*(f[q], \tilde{z}_1), \tag{3.2}
\]

\[
x\_2\_\^∗ ∈ X\^*(f[q], \tilde{z}_2). \tag{3.3}
\]

Then, it follows from (SC\_2\_G) that there exists some \(x ∈ X\^*(f[q], \tilde{z}_1)\) such that \(x \wedge \tilde{z}_2 ≤ x\_2\_\^∗\), implying

\[
\min\{x\_1\_\^∗(w), z\_2(w)\} = \min\{x(w), \tilde{z}_2(w)\} ≤ x\_2\_\^∗(w) \quad (w ∈ V \setminus S\+),
\]

where the equality is by (3.2) and the definition of \(\tilde{z}_2\). Hence, we have the desired inequality (3.1).

We now show that there exists a vector \(q ∈ R^V\) satisfying (3.2) and (3.3). Let \(k\) be a sufficiently large positive number such that \(k > \tilde{z}_1(w) - x\_1\_\^∗(w) (w ∈ S\+)\). Define \(d ∈ R^V\) by

\[
d(w) = \begin{cases} 
\frac{1}{k|S\+|} & (w ∈ S\+), \\
1 & (w ∈ V \setminus S\+). 
\end{cases}
\]

For \(i = 1, 2,\) we define a value \(η_i ∈ R\) by

\[
η_i = \max\{(d, x) \mid x ∈ X\^*(f[p], \tilde{z}_i)\}.
\]
Since the set $\tilde{Y}_i = \{y \in \mathbb{Z}^V \mid \langle d, y \rangle > \eta_i, \ y \geq \tilde{z}_i \}$ is finite and satisfies $f[p](y) > f[p](x^*_i) \ (y \in \tilde{Y}_i)$, we have

$$X^*(f[q], \tilde{z}_i) = \{x \mid x \in X^*(f[p], \tilde{z}_i), \ \langle d, x \rangle = \eta_i \} \quad (i = 1, 2) \tag{3.4}$$

by putting $q = p - \varepsilon d$ with a sufficiently small positive number $\varepsilon$.

To show that the condition (3.2) holds, let $x \in X^*(f[q], \tilde{z}_1)$. For $w \in V \setminus S^+$, we have $x(w) \leq \tilde{z}_1(w) = x^*_1(w)$, implying $x(V \setminus S^+) - x^*_1(V \setminus S^+) \leq 0$. By (3.4), we have

$$0 \leq \langle d, x \rangle - \langle d, x^*_1 \rangle = \frac{1}{k|S^+|} \sum_{w \in S^+} \{x(w) - x^*_1(w)\} + x(V \setminus S^+) - x^*_1(V \setminus S^+) \leq \frac{1}{k|S^+|} \sum_{w \in S^+} \{\tilde{z}_1(w) - x^*_1(w)\} + x(V \setminus S^+) - x^*_1(V \setminus S^+).$$

Since $(1/k|S^+|) \sum_{w \in S^+} \{\tilde{z}_1(w) - x^*_1(w)\} < 1$ and $x(V \setminus S^+) - x^*_1(V \setminus S^+)$ is a nonpositive integer, we have $x(V \setminus S^+) - x^*_1(V \setminus S^+) = 0$, implying (3.2).

We next prove that the condition (3.3) holds. It suffices to show that $\langle d, y \rangle \leq \langle d, x^*_2 \rangle$ for all $y \in X^*(f[p], \tilde{z}_2)$. By the definition of $\tilde{z}_2$, we have $y(S^+) \leq \tilde{z}_2(S^+) = x^*_2(S^+)$ and $y(w) \leq \tilde{z}_2(w) \leq x^*_1(w) \ (w \in V \setminus S^+)$, where the latter implies $\text{supp}^+(y-x^*_1) \subseteq S^+$. By the choice of $x^*_2$, it holds that $\text{supp}^+(y-x^*_1) = S^+$ and $y(V \setminus S^+) \leq x^*_2(V \setminus S^+)$. Therefore,

$$\langle d, y \rangle - \langle d, x^*_2 \rangle = \frac{y(S^+) - x^*_2(S^+)}{k|S^+|} + \{y(V \setminus S^+) - x^*_2(V \setminus S^+)\} \leq 0.$$

This concludes the proof of Theorem 3.2 (i).

### 3.2 Proof of “$(\text{SC}^1_G) \implies f \text{ is } M^2\text{-convex}”$

We prove Theorem 3.2 (ii).

Let $f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ be a function such that dom $f$ is bounded, and suppose that $f$ satisfies $(\text{SC}^1_G)$. We prove the $M^2$-convexity of $f$ by using Theorem 2.2, a characterization of $M^2$-convex functions by the sets of minimizers. Since $f[p]$ satisfies $(\text{SC}^1_G)$ for all $p \in \mathbb{R}^V$, it suffices to show that $\text{arg min} f$ is an $M^2$-convex set. To prove the $M^2$-convexity of arg min $f$, we use Theorem 2.1; we first consider the case where $x \leq y$ or $x \geq y$ (Lemma 3.3), then the case where $x - y = \chi_s + \chi_u - \chi_r - \chi_t$ for some $r, s, t, u \in V \cup \{0\}$ (Lemmas 3.4, 3.6, 3.7), and finally the general case (Lemma 3.9).

**Lemma 3.3.** For any $x, y \in \text{arg min} f$ with $x \leq y$, we have $[x, y]_Z \subseteq \text{arg min} f$.

**Proof.** We show that any $\tilde{x} \in [x, y]_Z$ is contained in arg min $f$. Since $y \in X^*(f, y)$ and $\tilde{x} \leq y$, $(\text{SC}^1_G)$ implies that there exists some $x_2 \in X^*(f, \tilde{x}) (\subseteq \text{arg min} f)$ such that $\tilde{x} = \tilde{x} \wedge y \leq x_2 \leq \tilde{x}$, i.e., $x_2 = \tilde{x}$. \(\blacksquare\)

**Lemma 3.4.** For any $x, y \in \text{arg min} f$ with $x - y = 2\chi_u - \chi_v$ for some distinct $u, v \in V$, we have $x - \chi_u, x - \chi_u + \chi_v \in \text{arg min} f$.

**Proof.** We firstly prove that $x - \chi_u + \chi_v \in \text{arg min} f$. If $x + \chi_v \in \text{arg min} f$, then Lemma 3.3 implies $x - \chi_u + \chi_v \in \text{arg min} f$ since $x - \chi_u + \chi_v \in [y, x + \chi_v]_Z$. Hence, we assume $x + \chi_v \not\in \text{arg min} f$. Let $M$ be a sufficiently large positive number, and $\varepsilon$ be a sufficiently small positive number. We define $p \in \mathbb{R}^V$ by

$$p(w) = \begin{cases} 
-2\varepsilon & \text{if } w = u, \\
-3\varepsilon & \text{if } w = v, \\
-M & \text{otherwise.}
\end{cases}$$


Assume, to the contrary, that \( x - \chi_u + \chi_v \not\in \arg\min f \). Then, we have \( X^*(f[p], x - \chi_u + \chi_v) = \{y\} \) and \( X^*(f[p], x + \chi_v) = \{x\} \). Since \( x - \chi_u + \chi_v \leq x + \chi_v \), it follows from (SC\(_1^V\)) that \( x - \chi_u = (x - \chi_u + \chi_v) \land x \leq y \), a contradiction since \( x(u) - 1 > y(u) \). Hence, \( x - \chi_u + \chi_v \in \arg\min f \) holds.

We then prove that \( x - \chi_u \in \arg\min f \). If there exists some \( x' \in \arg\min f \) with \( x' \leq x - \chi_u \), then Lemma 3.3 implies \( x - \chi_u \in \arg\min f \) since \( x - \chi_u \in [x', x] \mathbb{Z} \). Hence, we assume that there exists no such \( x' \in \arg\min f \), and derive a contradiction. Put \( x_s = x + \chi_v - \alpha_s \chi_v \) and \( y_s = x + \chi_v - \beta_s \chi_u \), where

\[
\alpha_s = \max\{\alpha \mid x + \chi_v - \alpha \chi_v \in \arg\min f\}, \quad \beta_s = \max\{\beta \mid x + \chi_v - \beta \chi_u \in \arg\min f\}.
\]

We define \( \hat{p} \in \mathbb{R}^V \) by

\[
\hat{p}(w) = \begin{cases} 
\varepsilon \alpha_s & \text{if } w = u, \\
\varepsilon (\beta_s + 1) & \text{if } w = v, \\
-M & \text{otherwise}.
\end{cases}
\]

Then, we have \( X^*(f[\hat{p}], x + \chi_v) = \{x_s\} \) and \( X^*(f[\hat{p}], x - \chi_u + \chi_v) = \{y_s\} \). By (SC\(_1^V\)), we have \( x_s - \chi_u = (x - \chi_u + \chi_v) \land x_s \leq y_s \), a contradiction since \( x_s(u) - 1 = x(u) - 1 > y(u) \geq y_s(u) \).

**Lemma 3.5.** Let \( x, y \in \arg\min f \) be any distinct vectors with \( x(V) \geq y(V) \). Suppose that there exists no \( z \in \arg\min f \) satisfying \( z \leq x \lor y \), \( \supp(x - z) \subseteq \supp(x - y) \), and \( z(V) > x(V) \). Then, for any \( u \in \supp(x - y) \) there exists \( v \in \supp^-(x - y) \cup \{0\} \) such that \( x - \chi_u + \chi_v \in \arg\min f \).

**Proof.** Let \( u \in \supp(x - y) \). Since \( x \in X^*(f, x \lor y) \), it follows from (SC\(_1^V\)) that there exists some \( x_2 \in X^*(f, (x \lor y) - \chi_u) \subseteq \arg\min f \) such that \( ((x \lor y) - \chi_u) \land x \leq x_2 \). This inequality implies

\[
x_2(u) = x(u) - 1, \quad x_2(w) = x(w) \quad (w \in V \setminus \supp^-(x - y) \cup \{u\})),
\]

from which follows \( x(V) \geq x_2(V) \geq x(V) - 1 \). Hence, \( x_2 = x - \chi_u + \chi_v \) holds for some \( v \in \supp^-(x - y) \cup \{0\} \).

**Lemma 3.6.** For any \( x, y \in \arg\min f \) with \( x - y = \chi_u + \chi_v - \chi_u - \chi_v \) for some distinct \( s, u, v \in V \), we have \( x - \chi_u + \chi_v, x - \chi_u \in \arg\min f \) or \( x - \chi_u + \chi_v, x - \chi_u \in \arg\min f \) (or both).

**Proof.** It suffices to show the following claims hold:

(a) \( x - \chi_u + \chi_v \in \arg\min f \) or \( x - \chi_u \in \arg\min f \),

(b) \( x - \chi_u + \chi_v \in \arg\min f \) or \( x - \chi_s \in \arg\min f \),

(c) \( x - \chi_s + \chi_v \in \arg\min f \) or \( x - \chi_u + \chi_v \in \arg\min f \),

(d) \( x - \chi_s \in \arg\min f \) or \( x - \chi_u \in \arg\min f \).

We firstly prove the claims (a) and (b). If \( x + \chi_v \in \arg\min f \), then Lemma 3.3 implies \( \{x - \chi_u + \chi_v, x - \chi_u \} \subseteq \{y, x + \chi_v\} \subseteq \arg\min f \). If \( x + \chi_v \not\in \arg\min f \), then Lemma 3.5 for \( x \) and \( y \) implies (a) and (b) since \( \supp^-(x - y) = \{v\} \).

We then prove (c). Assume, to the contrary, that neither \( x - \chi_s + \chi_v \) nor \( x - \chi_u + \chi_v \) is in \( \arg\min f \). Then, we have \( x - \chi_u \in \arg\min f \) by (a). Since \( x - \chi_u \leq x - \chi_u + \chi_v \leq x + \chi_v \), Lemma 3.3 implies \( x + \chi_v \not\in \arg\min f \). Put \( z_1 = x + \chi_v \) and \( z_2 = x - \chi_u + \chi_v \). Let \( M \) be a sufficiently large positive number, and \( \varepsilon \) be a sufficiently small positive number. We define \( p \in \mathbb{R}^V \) by

\[
p(w) = \begin{cases} 
-2\varepsilon & \text{if } w \in \{s, u\}, \\
-3\varepsilon & \text{if } w = v, \\
-M & \text{otherwise}.
\end{cases}
\]
Then, \( X^*(f[p], z_1) = \{ x \} \). By (SC\(_1^1\)), there exists some \( x_2 \in X^*(f[p], z_2) \) with \( x - \chi_u = z_2 \land x \leq x_2 \leq x - \chi_u + \chi_v \), i.e., \( x_2 \) is either \( x - \chi_u \) or \( x - \chi_u + \chi_v \). However, we have
\[
\begin{align*}
  f[p](x - \chi_u) - f[p](y) &= \varepsilon + f(x - \chi_u) - f(y) > 0, \\
  f[p](x - \chi_u + \chi_v) - f[p](y) &= -2\varepsilon + f(x - \chi_u + \chi_v) - f(y) > 0
\end{align*}
\]
since \( y \in \arg \min f \) and \( x - \chi_u + \chi_v \notin \arg \min f \). This shows that \( x_2 \notin X^*(f[p], z_2) \), a contradiction. Hence, the claim (c) holds.

We finally prove (d). Assume, to the contrary, that neither \( x - \chi_s \) nor \( x - \chi_u \) is in \( \arg \min f \). Since \( \{ x, x - \chi_u + \chi_v, x - \chi_u + \chi_v \} \subseteq \arg \min f \) by (a) and (b), Lemma 3.4 implies \( x - 2\chi_u + \chi_v, x - 2\chi_u + \chi_v \notin \arg \min f \). By Lemma 3.3, if \( x' \in \mathbb{Z}^V \) satisfies at least one of the inequalities \( x' \leq x - \chi_u, x' \leq x - \chi_s, x' \leq x - 2\chi_u + \chi_v \), and \( x' \leq x - 2\chi_s + \chi_v \), then \( x' \notin \arg \min f \). This shows that \( \arg \min f \cap \{ x' \mid x' \leq z_1 \} \subseteq \{ x, y, x - \chi_u + \chi_v, x - \chi_s + \chi_v, x + \chi_v \} \), where \( z_1 = x + \chi_v \). We define \( \widehat{p} \in \mathbb{R}^V \) by
\[
\widehat{p}(w) = \begin{cases} 
\varepsilon & \text{if } w \in \{ s, u \}, \\
3\varepsilon & \text{if } w = v, \\
-M & \text{otherwise}.
\end{cases}
\]
Then, we have \( X^*(f[\widehat{p}], z_1) = \{ x \} \) and \( X^*(f[\widehat{p}], z_2) = \{ y \} \), where \( z_2 = x - \chi_u + \chi_v \). By (SC\(_1^1\)), we have \( x - \chi_u = z_2 \land x \leq y \), a contradiction since \( x(s) > y(s) \). Hence, the claim (d) holds.

Lemma 3.7. Let \( x, y \in \text{dom } f \) be any vectors satisfying \( ||x - y||_1 = 4 \) and \( x(V) = y(V) \), and \( u \in \text{supp}(x - y) \). Then, there exist \( v, w \in \text{supp}^+(x - y) \cap \{ 0 \} \) such that \( x - \chi_u + \chi_v, y + \chi_u - \chi_w \in \arg \min f \).

Proof. Suppose that \( y = x - \chi_s - \chi_u + \chi_r + \chi_t \) for some \( r, s, t, u \in V \) with \( \{ s, u \} \cap \{ r, t \} = \emptyset \). We show that \( x - \chi_u + \chi_v \in \arg \min f \) and \( y + \chi_u - \chi_w \in \arg \min f \) for some \( v, w \in \{ r, t, 0 \} \).

We firstly consider the case where there exists some \( z \in \arg \min f \) satisfying
\[
z \leq x \lor y, \quad \text{supp}(x - z) \subseteq \text{supp}(x - y), \quad z(V) > x(V).
\]
This assumption implies
\[
\{ x + \chi_r, x + \chi_t, x + \chi_r + \chi_t, y + \chi_s, y + \chi_u \} \cap \arg \min f \neq \emptyset.
\]
We first claim that \( x + \chi_r \in \arg \min f \) or \( x + \chi_t \in \arg \min f \) holds. If \( x + \chi_r + \chi_t \in \arg \min f \), then Lemma 3.3 implies \( \{ x + \chi_r, x + \chi_t \} \subseteq \arg \min f \). If \( y + \chi_u \in \arg \min f \), then Lemmas 3.4 and 3.6 for \( y + \chi_u = x - \chi_s + \chi_u + \chi_t \) and \( x \) imply \( x + \chi_r \in \arg \min f \) or \( x + \chi_t \in \arg \min f \). The case where \( y + \chi_u \in \arg \min f \) can be dealt with similarly.

We, w.l.o.g., assume that \( x + \chi_r \in \arg \min f \). Lemmas 3.4 and 3.6 for \( x + \chi_r = y + \chi_u + \chi_s - \chi_t \) and \( y \) imply \( \{ y + \chi_u, y + \chi_s - \chi_t \} \subseteq \arg \min f \) or \( \{ y + \chi_s, y + \chi_u - \chi_t \} \subseteq \arg \min f \). If the former holds, then we are done since \( y + \chi_s - \chi_t = x - \chi_u + \chi_r \). If the latter holds, then we can apply Lemmas 3.4 and 3.6 to \( y + \chi_s = x - \chi_u + \chi_r + \chi_t \) and \( x \) to obtain \( x - \chi_u + \chi_r \in \arg \min f \) or \( x - \chi_u + \chi_t \in \arg \min f \).

We then consider the case where there exists no \( z \in \arg \min f \) satisfying (3.5). By Lemma 3.5, we have \( x - \chi_u + \chi_v \in \arg \min f \) and \( x - \chi_s + \chi_v \in \arg \min f \) for some \( v, v' \in \{ r, t, 0 \} \). If \( v' \neq 0 \), then we have \( x - \chi_s + \chi_v' = y + \chi_u - \chi_w \) for some \( w \in \{ r, t \} \). If \( v' = 0 \), then we can apply Lemmas 3.4 and 3.6 to \( y \) and \( x - \chi_s \) to obtain \( y + \chi_u - \chi_r \in \arg \min f \) or \( y + \chi_u - \chi_t \in \arg \min f \).

Lemma 3.8. Let \( x, y, z \in \mathbb{Z}^V \) be any distinct vectors with \( z \leq x \lor y \) and \( z(V) > \max \{ x(V), y(V) \} \). Then, we have \( ||z - x||_1 < ||x - y||_1 \) and \( ||z - y||_1 < ||x - y||_1 \).
Proof. We prove $|z-x|_1 < |x-y|_1$ only. Put $S^+ = \text{supp}^+ (x-y)$, $C = \text{supp}^- (x-z) \subseteq \text{supp}^- (x-y)$, $D = \text{supp}^- (x-y) \setminus C$, and $E = V \setminus \text{supp}(x-y)$. Then,

$$
|x-y|_1 - |x-z|_1 = z(S^+ \cup D \cup E) + y(C \cup D) - y(S^+) - z(C) - 2x(D) - x(E) 
$$

$$
> 2[y(C) - z(C)] + 2[y(D) - x(D)] \geq 0,
$$

where the first inequality is by $z(V) > y(V)$ and $y(E) = x(E)$, and the second by $y(C) \geq z(C)$ and $y(D) \geq x(D)$.

Lemma 3.9. arg min $f$ satisfies (B$^3$-EXC$_\pm$), i.e., arg min $f$ is an $M$-convex set if it is nonempty.

Proof. Let $x, y \in \text{arg min } f$ and $u \in \text{supp}^+ (x-y)$. We show by induction on $|x-y|_1$ that

$$
x - \chi_u + \chi_v \in \text{arg min } f \quad (\exists v \in \text{supp}^+ (x-y) \cup \{0\}), \quad (3.6)$$

$$
y + \chi_u - \chi_w \in \text{arg min } f \quad (\exists w \in \text{supp}^- (x-y) \cup \{0\}). \quad (3.7)
$$

By Lemmas 3.3, 3.4, and 3.6, we may assume $\text{supp}^+ (x-y) \neq \emptyset$, $\text{supp}^- (x-y) \neq \emptyset$, and $|x-y|_1 \geq 4$.

We first claim that the following (3.8) or (3.9) holds:

$$
x' = x - \chi_s + \chi_t \in \text{arg min } f \quad (\exists s \in \text{supp}^+ (x-y), \exists t \in \text{supp}^- (x-y) \cup \{0\}), \quad (3.8)$$

$$
y' = y + \chi_s - \chi_j \in \text{arg min } f \quad (\exists i \in \text{supp}^+ (x-y) \cup \{0\}, \exists j \in \text{supp}^- (x-y)). \quad (3.9)
$$

If there exists no $z \in \text{arg min } f$ satisfying $z \leq x \vee y, \text{supp}(x-z) \subseteq \text{supp}(x-y)$, and $z(V) > \max \{x(V), y(V)\}$, then Lemma 3.5 implies (3.8) or (3.9) according as $x(V) \geq y(V)$ or $x(V) < y(V)$. Hence, we assume that such $z \in \text{arg min } f$ exists. We may also assume $z \neq x \vee y$, since otherwise $(x \vee y) - \chi_w \in \text{arg min } f (\forall w \in \text{supp}(x-y))$ holds by Lemma 3.3. Therefore, we have $\text{supp}^+ (x-z) \cap \text{supp}^+ (x-y) \neq \emptyset$ or $\text{supp}^- (z-y) \cap \text{supp}^- (x-y) \neq \emptyset$. Note that $|x-z|_1 < |x-y|_1$ and $|y-z|_1 < |x-y|_1$ by Lemma 3.8. If $\text{supp}(x-z) \cap \text{supp}^+ (x-y) \neq \emptyset$, then the induction hypothesis for $x$ and $z$ implies $x - \chi_s + \chi_t \in \text{arg min } f$ for some $s \in \text{supp}^+ (x-z) \cap \text{supp}^+ (x-y)$ and $t \in \text{supp}^- (x-z) \cup \{0\} \subseteq \text{supp}^- (x-y) \cup \{0\}$, i.e., (3.8) holds. Similarly, (3.9) holds if $\text{supp}^- (z-y) \cap \text{supp}^- (x-y) \neq \emptyset$.

In the following, we assume that (3.8) holds; the case where (3.9) holds can be dealt with similarly and therefore the proof is omitted.

Case 1: $\text{supp}^+ (x-y) = \emptyset$ We have $\text{supp}^+ (x-y) = \{u\}$, implying $x' = x - \chi_u + \chi_t \in \text{arg min } f$ for $j \in \text{supp}^- (x-y) \subseteq \text{supp}^- (x-y)$. Since $|x - (y - \chi_j)| < |x - y|_1$ and $\text{supp}^+ (x - (y - \chi_j)) = \{u\}$, the induction hypothesis implies $(y - \chi_j) + \chi_u - \chi_h \in \text{arg min } f$ for some $h \in \text{supp}^- (x - (y - \chi_j)) \cup \{0\} \subseteq \text{supp}^- (x-y) \cup \{0\}$. If $h \neq 0$ then we apply Lemma 3.4 or 3.6 to $y - \chi_j + \chi_u - \chi_h$ and $y$ to obtain $\{y + \chi_u - \chi_j, y + \chi_u - \chi_h\} \cap \text{arg min } f \neq \emptyset$, i.e., (3.7) holds.

Case 2: $\text{supp}^+ (x-y) \neq \emptyset, u \not\in \text{supp}^+ (x-y)$ Since $u \in \text{supp}^+ (x-y)$, we have $x' = x - \chi_u + \chi_t$ for some $t \in \text{supp}^- (x-y) \cup \{0\}$, i.e., (3.6) holds. Since $|x' - y|_1 < |x - y|_1$, the induction hypothesis for $x'$ and $y$ implies $\tilde{y} = y + \chi_i - \chi_j \in \text{arg min } f$ for some $i \in \text{supp}^+ (x'-y) \subseteq \text{supp}^+ (x-y) \setminus \{u\}$ and $j \in \text{supp}^- (x'-y) \cup \{0\} \subseteq \text{supp}^- (x-y) \cup \{0\}$. Since $|x - \tilde{y}|_1 < |x - y|_1$, the induction hypothesis for $x$, $\tilde{y}$, and $u \in \text{supp}^+ (x - \tilde{y})$ implies $\tilde{y} + \chi_u - \chi_h \in \text{arg min } f$ for some $h \in \text{supp}^- (x - \tilde{y}) \cup \{0\} \subseteq \text{supp}^- (x-y) \cup \{0\}$. Applying Lemma 3.3, 3.4, 3.6, or 3.7 to $\tilde{y} + \chi_u - \chi_h = y + \chi_i + \chi_u - \chi_j - \chi_h$ and $y$, we have $\{y + \chi_u - \chi_j, y + \chi_u - \chi_h\} \cap \text{arg min } f \neq \emptyset$, i.e., (3.7) holds.

Case 3: $u \in \text{supp}^+ (x'-y)$ Since $|x' - y|_1 < |x - y|_1$, the induction hypothesis for $x'$, $y$, and $u \in \text{supp}^+ (x'-y)$ implies $y + \chi_u - \chi_w \in \text{arg min } f$ for some $w \in \text{supp}^- (x'-y) \cup \{0\} \subseteq \text{supp}^- (x-y) \cup \{0\}$, i.e., (3.7) holds. By using this fact we can show (3.6) in a similar way as in Case 2.

□
4 Concluding Remarks

It is shown in [3, 5, 6] that $M^2$-convexity of a function $f : \mathbb{Z}^V \to R \cup \{+\infty\}$ implies the properties (SC$^1$) and (SC$^2$). Theorem 3.1 is an immediate consequence of this fact since $f|_p$ is $M^2$-convex for any $p \in R^V$ if $f$ is $M^2$-convex. In fact, the properties (SC$^1$) and (SC$^2$) hold true under a weaker assumption than $M^2$-convexity. We call a function $f$ *semistrictly quasi $M^2$-convex* if dom $f \neq \emptyset$ and it satisfies (SSQM$^F$):

$$(SSQM^F) \forall x, y \in \text{dom } f, \forall u \in \text{supp}^+(x - y), \exists v \in \text{supp}^-(x - y) \cup \{0\}:
(i) f(x - \chi_u + \chi_v) \geq f(x) \implies f(y + \chi_u - \chi_v) \leq f(y), \quad \text{and}
(ii) f(y + \chi_u - \chi_v) \geq f(y) \implies f(x - \chi_u + \chi_v) \leq f(x).$$

It is easy to see that any $M^2$-convex function satisfies (SSQM$^F$). See [12] for more accounts on semistrictly quasi $M^2$-convex functions.

**Theorem 4.1.** A function $f : \mathbb{Z}^V \to R \cup \{+\infty\}$ with (SSQM$^F$) satisfies (SC$^1$) and (SC$^2$).

**Proof.** We prove (SC$^1$) only; (SC$^2$) can be shown similarly and the proof is omitted.

Let $x_1, x_2 \in \mathbb{Z}^V$ be any vectors with $x_1 \geq z_2$ and $X^*(f, z_2) \neq \emptyset$. Also, let $x_1 \in X^*(f, z_1)$. We choose $x_2 \in X^*(f, z_2)$ minimizing the value $\sum\{x_1(w) - x_2(w) \mid w \in \text{supp}^+(x_1 \wedge z_2 - x_2)\}$. Assume, to the contrary, that $\text{supp}^+(x_1 \wedge z_2 - x_2) \neq \emptyset$. Let $u \in \text{supp}^+(x_1 \wedge z_2 - x_2) \subseteq \text{supp}^+(x_1 - x_2)$. By (SSQM$^F$), there exists $v \in \text{supp}^-(x_1 - x_2) \cup \{0\}$ such that if $f(x_1 - \chi_u + \chi_v) \geq f(x_1)$ then $f(x_2 + \chi_u - \chi_v) \leq f(x_2)$. Since $x_1 - \chi_u + \chi_v \leq x_1 \vee x_2 \leq z_1$, we have $f(x_1 - \chi_u + \chi_v) \geq f(x_1)$. Hence, $f(x_2 + \chi_u - \chi_v) \leq f(x_2)$ follows. By the choice of $u$ we have $x_2 + \chi_u - \chi_v \leq z_2$. This implies that $x_2 + \chi_u - \chi_v \in X^*(f, z_2)$, which contradicts the choice of $x_2$. Hence we have $x_1 \wedge z_2 \leq x_2$. \hfill \square

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**References**


