Efficiently Pricing European-Asian Options — Ultimate Implementation and Analysis of the AMO Algorithm —

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Abstract. We propose an efficient and accurate randomized approximation algorithm for pricing a European-Asian option on the binomial tree model. For an option with the strike price X on an n-step binomial tree and any positive integer k, our algorithm runs in $\mathrm{O}(kn^2)$ time with the error bound $\mathrm{O}(X/k)$ which is independent of n. Our algorithm is a modification of the approximation algorithm developed by Aingworth, Motwani, and Oldham (2000) into a randomized algorithm, which improves the accuracy theoretically as well as practically.

1 Introduction

1.1 Background

Options are popular financial instruments in world financial markets. One of the simplest options is *European call* option, which is a contract giving its holder the right, but not the obligation, to buy a stock or other financial asset at some point in the future (called the *expiration date*) for a specified price X (called the *strike price*). The *payoff* of an option is the amount of money its holder makes on the contract. Suppose that we have a European option on a stock, and the stock price S is more than the strike price X on the expiration date. Then, we can make some money by *exercising* the option to buy the stock and selling the stock immediately at the market price. Hence, the payoff of a European option is given by $(S - X)^+ = \max\{S - X, 0\}$. The price of the option is usually much less than the actual price of the underlying stock. Therefore, options hedge risk more cheaply than stocks only, and provide a chance to get large profit with a small amount of money if one's speculation is good.

The price of an option is given by the discounted expected value of the payoff. Because of the popularity of options, techniques for computing the option price have extensively been discussed in the literature [1, 2, 5–11, 13]. A standard method of pricing an option is to model the movement of the underlying financial asset as geometric Brownian motion with drift and then to construct an arbitrage portfolio [4, 11]. This yields a stochastic differential equation, and its solution gives the option price. However, it is often difficult to solve this differential equation for many complex options such as European-Asian option dealt with in this paper, and indeed no simple closed-form solution is known. Therefore, it is widely practiced to simulate geometric Brownian motion by using a discrete model, and use this model to approximate the option price. One such discrete model is the *binomial tree model* [8, 11], where the time period is decomposed into n time steps, and geometric Brownian motion is modeled by using a biased random walk on a graph called a *binomial tree* of depth n. The option price obtained from the binomial tree model converges to the price given by the differential equation if n goes to infinity. In the binomial tree model, the process of the movement of a stock price is represented by a path. An option is said to be *path-dependent* [6, 11] if the option's payoff depends on the path representing the process as well as the current stock price. Although path-dependency is often useful in designing a secure option against risk caused by sudden change of the market, it makes the analysis of option prices quite difficult.

1.2 Our Problem and Result

In this paper, we consider the pricing of *European-Asian* option. European-Asian option is a kind of path-dependent options and its payoff is given as $(A - X)^+$, where A is the average stock price during the time from the purchase date to the expiration date of the option and X is the strike price. It is known to be #P-hard in general to compute the exact price of path-dependent options on the binomial tree model [6]. Therefore, it is desired to design an efficient approximation algorithm with provable high accuracy, and various pricing techniques have been developed so far [1, 2, 6, 7, 9, 13].

A naive method for computing the exact price of European-Asian options, called the full-path method, enumerates all paths in the binomial tree. Unfortunately, the full-path method requires exponential time since there are exponential number of paths in the binomial tree. Hence, the Monte Carlo method that samples paths in the binomial tree is popularly used to compute an approximate value of the exact price. The error bound of the Monte Carlo method, however, depends on the volatility of the stock price when a polynomial number of samples are taken by naive sampling [10].

Aingworth–Motwani–Oldham (AMO) [1] proposed the first polynomial-time approximation algorithm with guaranteed worst-case error bound, which enables us to avoid the influence of volatility to the theoretical error bound. The idea is to prune exponential number of high-payoff paths by using mathematical formulae during the run of an aggregation algorithm based on dynamic programming and bucketing. In each of n aggregation steps the algorithm produces the error bounded by X/k, where k denotes the number of buckets used at each node of the binomial tree. Hence, the error bound of the AMO algorithm is nX/k, and the algorithm runs in $\mathrm{O}(kn^2)$ time. While algorithms on the "uniform" model has been mainly considered in the literature [1, 2, 6, 7, 9], the AMO algorithm and its analysis work on the "non-uniform" model where the transition probabilities of the stock price may differ at each node [13].

Then, variants of the AMO algorithm were proposed to achieve a better error bound than nX/k. Akcoglu–Kao–Raghavan [2] presented a recursive version of the AMO algorithm and reduce the error bound to $\mathrm{O}(n^{\frac{1+\varepsilon}{2}} X/k)$ by spending almost the same time complexity when the volatility of the stock is small.

The error bound is further improved by Dai–Huang–Lyuu (DHL) [9] and by Ohta–Sadakane–Shioura–Tokuyama (OSST) [13]. While the AMO algorithm uses the same number of buckets at each node of the binomial tree, the DHL algorithm [9] uses different number of buckets at each node. By adjusting the number of buckets at each node appropriately while keeping the time complexity $\mathrm{O}(kn^2)$, they achieved the error bound $\mathrm{O}(\sqrt{n}X/k)$, where k is the average number of buckets used at each node. Their analysis, however, applies only to the uniform model and does not extend to the non-uniform model. On the other hand, the OSST algorithm [13] uses the idea of randomized rounding in the aggregation steps of the algorithm, and achieves the error bound $\mathrm{O}(\sqrt{n}X/k)$ for the non-uniform model. Moreover, it is shown in [13] that for the uniform model the error bound of the OSST algorithm can be reduced to $\mathrm{O}(n^{1/4}X/k)$.

In this paper, we further reduce the error bound by giving a randomized approximation algorithm with an $\mathrm{O}(kn^2)$ time complexity and an $\mathrm{O}(X/k)$ error bound. The error bound of our algorithm is independent of the depth n of the binomial tree, although those of the AMO algorithm and its previous variants [2, 9, 13] are dependent on n. Our algorithm uses the ideas in Dai et al. [9] and Ohta et al. [13]. As in [13], we regard the aggregation steps of the algorithm as a Martingale process with $\mathrm{O}(n^2)$ random steps by using novel random variables. It can be shown that the expected value of the output by our algorithm equals the exact price, and that the error in each single step is bounded by a function of the number of buckets at a node of the binomial tree. Thus, we can apply Azuma's inequality [3] to the Martingale process to obtain the error bound. If we choose k as the number of buckets at each node, the algorithm coincides with the one in [13]. To reduce the error bound as much as possible, we adjust the number of buckets at each node and obtain the error bound $\mathrm{O}(X/k)$, where k is the average number of buckets used at each node. Since the value X/k can be seen as the "average" of the absolute error produced at each node of the binomial tree, the error bound of our algorithm is the best possible within the framework of the AMO algorithm. We also show the practical quality of the approximate value computed by our algorithm by some numerical experiments.

Although we only consider the pricing of call options on the binomial tree model in this paper, our algorithm can be easily modified to put options and to the trinomial tree model as in [1, 2, 9, 13].

2 Preliminaries

2.1 The Binomial Tree Model

A *binomial tree* of depth n is a leveled directed acyclic graph defined as follows (see Fig. 1). A binomial tree of depth n has $n+1$ levels. There are $i+1$ nodes in the i-th level $(0 \leq i \leq n)$ and each node is labeled as (i, j), where j $(0 \leq j \leq i)$ denotes the numbering of the nodes. The node $(0, 0)$ in the 0-th level is called

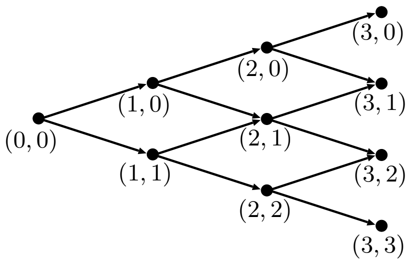


Fig. 1. A binomial tree of depth 3

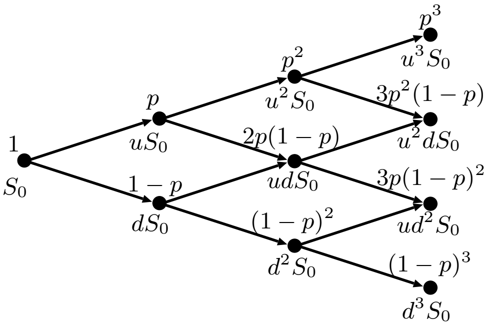


Fig. 2. The uniform binomial tree model. The probability $\omega(i, j)$ (resp., the stock price $S_i(j)$) at each node is shown above the node (resp., below the node).

the *root*, and each node (n, j) in the n-th level is called a *leaf*. Each non-leaf node (i, j) has two children $(i+1, j)$ and $(i+1, j+1)$. Therefore, each non-root node (i, j) has two parents $(i-1, j-1)$ and $(i-1, j)$ if $1 \le j \le i-1$, and each of $(i, 0)$ and (i, i) has only one parent.

Let us consider a discrete random process simulating the movement of a stock price. We divide the time from the purchase date to the expiration date of an option into n time periods, and the i-th time step means the end of the i-th time period. In particular, 0-th (resp., n-th) time step is the purchase (resp., expiration) date of the option. For $i = 0, 1, \ldots, n$, let S_i be a random variable representing the stock price at the i-th time step, where S_0 is the initial stock price known in advance. The fundamental assumption in the binomial tree model is that in each time step the stock price S either rises to uS or falls to dS, where u and d are predetermined constants satisfying $u > d$ and $u = 1/d$. Thus, we can model the stock price movement by using a binomial tree (see Fig. 2).

Suppose that we are at a non-leaf node (i, j) in the binomial tree model and the current stock price is S. With probability p_{ij}, we move to the node $(i+1, j)$ and the stock price rises to uS; with probability $1 - p_{ij}$, we move to the node $(i+1, j+1)$ and the stock price falls to dS. Thus, the stock price at the node (i, j) is $S_i(j) = u^{i-j} d^j S_0$. The binomial tree model is said to be *uniform* if $p_{ij} = p$ for each node (i, j); otherwise it is *non-uniform*. The uniform model has been widely considered $[1, 2, 6, 7, 9]$ since p is uniquely determined under the non-arbitrage condition of the underlying stock. The non-uniform model, however, is often useful to deal with various stochastic models. For each node (i, j), we denote by $\omega(i, j)$ the probability that the random walk reaches to (i, j). In the uniform model, we have $\omega(i, j) = \binom{i}{j} p^{i-j}(1-p)^j$.

2.2 Options

Let X be the strike price of an option. The *payoff* of an option is the amount of money its holder makes on the contract. We adopt a convention to write F^+ for the value $\max\{F, 0\}$. *European option* is one of basic options, and its payoff is given by $(S_n - X)^+$ which is determined by the stock price S_n on the expiration date. It is quite easy to compute the expected value of the payoff of European options under the binomial tree model. A drawback of European options is that the payoff may be affected drastically by the movement of the stock price just before the expiration date; even if the stock price goes very high during most of time periods, it may happen that the option does not make money at the end. *European-Asian option* is more reliable for the holder than European option, and its payoff is given by $(A_n - X)^+$, where $A_n = (\sum_{i=0}^{n} S_i)/(n+1)$ is the average of the stock prices during n time periods. Let $T_j = \sum_{i=0}^{j} S_i$ be the *running total* of the stock price up to the j-th time step. Once T_j exceeds the threshold $(n+1)X$, the option holder will surely exercise it on the expiration date and obtain the payoff of at least $T_j/(n+1) - X$.

Our aim is to compute the price of European-Asian options. Since the price of an option is given by the discounted expected value of the payoff, it suffices to compute the expected payoff. A simple method is to compute the running total $T_n(\mathcal{P})$ of the stock price for each path $\mathcal{P}$ in the binomial tree together with the probability $\Pr(\mathcal{P})$ that the path occurs, and exactly compute the value

$$E((A_n - X)^+) = \sum\{\Pr(\mathcal{P}) \cdot (\tfrac{T_n(\mathcal{P})}{n+1} - X)^+ \mid \mathcal{P} : \text{ a path from the root to a leaf}\}.$$

We call the expected value of the payoff computed as above the *exact value* of the expected payoff, and denote $U = E((A_n - X)^+)$. This simple method, however, needs exponential time since there are 2^n paths in a binomial tree.

3 A New Algorithm for Pricing European-Asian Options

3.1 A Basic Algorithm

We describe a basic approximation algorithm for the option's expected payoff. This algorithm is a slight generalization of the AMO algorithm, and the previous approximation algorithms in [1, 9, 13] can be seen as specialized versions of this basic algorithm.

As in [1], the basic algorithm uses dynamic programming to compute an approximate value of the option's expected payoff. For a path $\mathcal{P}$ from the root to a node (i, j) in the i-th level, we define the *state* of $\mathcal{P}$ as a pair $(S_i(j), T_i)$ of the stock price $S_i(j) = u^{i-j}d^j S_0$ and the running total T_i. Note that the states of two different paths reaching a node (i, j) can be the same. We define the *weight* of the state $(S_i(j), T_i)$ as the probability that a path $\mathcal{P}$ with the state $(S_i(j), T_i)$ occurs. The basic algorithm is based on a simple observation that if the running total of a current state is above the threshold $(n + 1)X$, then the conditional expectation of the payoff at this state can be analytically computed as shown in Lemma 1 below, and such a state can be pruned away.

Lemma 1 ([1, 13]). *Suppose that we are at a node (i,j) in the i-th level and the current state is (S,T), where $T \geq (n+1)X$. Then, the payoff's conditional expectation is given as $\{T + h(i,j)\}/(n+1) - X$, where $h(i,j)$ is defined by the following recursive formula: if $i = n$ then $h(i,j) = 0$, and if $i < n$ then*

$$h(i,j) = p_{ij}\{h(i+1,j) + S_{i+1}(j)\} + (1-p_{ij})\{h(i+1,j+1) + S_{i+1}(j+1)\}.$$

Hence, we need to consider only the states with running total less than $(n+1)X$, which may be exponential many. Rather than dealing with each unpruned state individually, we instead aggregate the states by using buckets that divide the interval $[0,(n+1)X)$. At each node (i,j) in the i-th level, the algorithm creates k_{ij} buckets $B_i(j,h)$ $(h = 0,1,\dots,k_{ij}-1)$, each of which corresponds to the interval $[b_h, b_{h+1}) = [\frac{(n+1)X}{k_{ij}}h, \frac{(n+1)X}{k_{ij}}(h+1))$. Each unpruned state of a path terminating at the node (i,j) is stored in one of k_{ij} buckets according to its running total. The algorithm chooses a value $R_i(j,h)$ in the interval $[b_h, b_{h+1})$ appropriately, and approximates all states in the bucket $B_i(j,h)$ by a single state $(S_i(j), R_i(j,h))$, where its weight $w_i(j,h)$ is given by the sum of the weights of all states in $B_i(j,h)$. Then, the algorithm produces two new states $(S_{i+1}(j), R_i(j,h) + S_{i+1}(j))$ and $(S_{i+1}(j+1), R_i(j,h) + S_{i+1}(j+1))$ in the $(i+1)$-st level, and inserts these states in appropriate buckets at the nodes $(i+1,j)$ and $(i+1,j+1)$, respectively, or computes the conditional expectation of the payoff at these states by using Lemma 1.

The error bound and the time and space complexity of the basic algorithm can be analyzed as follows.

Theorem 2. *The basic algorithm computes a value Ψ satisfying $|\Psi - U| \leq X\sum_{i=0}^{n}\sum_{j=0}^{i}\omega(i,j)/k_{ij}$. The time and space complexity of the basic algorithm are $\mathrm{O}(\sum_{i=0}^{n}\sum_{j=0}^{i}k_{ij})$.*

Proof. The error obtained by rounding a running total at a node (i,j) is bounded by $(n+1)X/k_{ij}$. Therefore, the contribution in processing one node to the error of the average stock value A_n is at most $X\omega(i,j)/k_{ij}$, and the error in the estimation of A_n is bounded by $X\sum_{i=0}^{n}\sum_{j=0}^{i}\omega(i,j)/k_{ij}$.

Let $b_i(j,h)$ be the number of states inserted in the bucket $B_i(j,h)$. Then, the time and space complexity are written as $\mathrm{O}(\sum_{i=0}^{n}\sum_{j=0}^{i}k_{ij} + \sum_{i=1}^{n}\sum_{j=0}^{i}b_i(j,h))$. Since we obtain from each bucket in the i-th level at most two new states in the $(i+1)$-st level, it holds that $\sum_{j=0}^{i+1}b_{i+1}(j,h) \leq 2\sum_{j=0}^{i}k_{ij}$ for $i = 0,\dots,n-1$. □

3.2 Previous Algorithms

We can obtain the algorithms [1, 9, 13] by customizing the number of buckets k_{ij} and the value $R_i(j,h)$.

The AMO algorithm [1] can be obtained by setting $k_{ij} = k$ with a positive integer k for all nodes (i,j) and $R_i(j,h) = \frac{(n+1)X}{k}h$. Note that the AMO algorithm computes a lower bound of the exact value U of the expected payoff; we can also compute an upper bound by setting $R_i(j,h) = \frac{(n+1)X}{k}(h+1)$ instead.

We denote by AMO-LB (resp., AMO-UB) the AMO algorithm for a lower (resp., upper) bound of U.

Dai et al. [9] proposed four approximation algorithms nUnifDown, nUnifCvg, nUnifUp, and nUnifSpl, where the first two (resp., the last two) computes lower (resp., upper) bounds of U. All algorithms use k_{ij} values defined as follows:

$$k_{ij} = \left\lceil \frac{k(n+1)(n+2)}{2} \times \frac{\sqrt{\omega(i,j)}}{\sum_{i'=0}^{n} \sum_{j'=0}^{i'} \sqrt{\omega(i',j')}} \right\rceil \qquad \text{for all nodes } (i,j),$$

where k is a positive integer corresponding to the average number of buckets at each node. In nUnifDown (resp., nUnifUp) we set $R_i(j,h) = \frac{(n+1)X}{k_{ij}} h$ (resp., $R_i(j,h) = \frac{(n+1)X}{k_{ij}}(h+1)$). The algorithms nUnifCvg and nUnifSpl are modified versions of nUnifDown and nUnifUp by using heuristics; see [9] for details. While the error bounds of nUnifCvg and nUnifSpl are the same as those of nUnifDown and nUnifUp theoretically, they are much better practically.

The OSST algorithm [13] is a randomized algorithm. We set $k_{ij} = k$ with a positive integer k for all nodes (i,j), as in the AMO algorithm. To set the value $R_i(j,h)$, we choose a "representative" state $(S_i(j), T)$ in the bucket $B_i(j,h)$ randomly, where a state with weight w is chosen with probability $w/w_i(j,h)$, and set $R_i(j,h) = T$.

3.3 Our Algorithm and Analysis

Our algorithm is based on the ideas used in Dai et al. [9] and Ohta et al. [13], and can be obtained from the basic algorithm. We set $R_i(j,h)$ in the same way as in the OSST algorithm, i.e., we choose a representative state $(S_i(j), T)$ in the bucket $B_i(j,h)$ randomly, where a state with weight w is chosen with probability $w/w_i(j,h)$, and set $R_i(j,h) = T$. We explain later how to choose k_{ij}.

Let Ψ be the payoff value computed by our algorithm. Since our algorithm is randomized, Ψ is a random variable depending on the random choice of representative states in the buckets. Let $Y_{i,j}$ be a random variable giving the future value of the payoff just after the algorithm processes the node (i,j) in the i-th level, i.e., after the choice of representatives in all buckets has been determined up to the j-th node in the i-th level. By definition, $Y_{0,0} = U$ and $Y_{n,n} = \Psi$. Thus, we have a random process with $\sum_{i=0}^{n}(i+1) = (n+1)(n+2)/2$ steps. The following lemma shows that random variables $Y_{0,0}, Y_{1,0}, \ldots, Y_{n,n}$ constitute a *Martingale sequence*.

Lemma 3. *$E(Y_{i,j} \mid Y_{0,0}, Y_{1,0}, Y_{1,1}, \ldots, Y_{i,j-1}) = Y_{i,j-1}$ for $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, i$.*

Proof. Consider the set $\{a_1, a_2, \ldots, a_q\}$ of states in a bucket at the node (i,j) of the i-th level before selecting a representative. For $l = 1, 2, \ldots, q$, let $Y(a_l)$ be the expected payoff (exactly computed from the model) for a path with the state a_l, and $w(a_l)$ be the weight of a_l. If the state a_l is selected, it contributes $Y(a_l)W$ to the payoff, where $W = \sum_{l=1}^{q} w(a_l)$. Thus, the expected contribution of the states after the selection is $\sum_{l=1}^{q}(w(a_l)/W)Y(a_l)W = \sum_{l=1}^{q} w(a_l)Y(a_l)$, where the right-hand side is the expected contribution before the selection. $\square$

Lemma 3 also shows that the expected value of the payoff Ψ equals the exact value U of the expected payoff, i.e., $E(Y_{n,n}) = E(\Psi) = U$.

When the algorithm processes a node (i,j), running totals of paths terminating at (i,j) are approximated with the error less than $(n+1)X/k_{ij}$, and the running totals of other paths remain the same. Hence, we have

$$\left.\begin{aligned} |Y_{i,j+1} - Y_{i,j}| &< \tfrac{X\omega(i,j+1)}{k_{i,j+1}} \ (0 \le j < i \le n), \\ |Y_{i+1,0} - Y_{i,i}| &< \tfrac{X\omega(i+1,0)}{k_{i+1,0}} \ (0 \le i < n). \end{aligned}\right\} \tag{1}$$

Thus, Azuma's inequality [3] applies (see also [12, Theorem 4.16]).

Theorem 4 (Azuma's inequality). *Let $Z_0, Z_1, \ldots$ be a Martingale sequence such that $|Z_k - Z_{k-1}| < c_k$ for each k, where c_k is a constant. Then,*

$$\Pr[|Z_t - Z_0| \ge \lambda] \le 2\exp\left(\frac{-\lambda^2}{2\sum_{k=1}^{t} c_k^2}\right) \qquad (\forall t = 1, 2, \ldots,\ \forall \lambda > 0).$$

Theorem 4 and (1) yield the inequality

$$\Pr[|Y_{n,n} - U| \ge \lambda] \le 2\exp\left(\frac{-\lambda^2}{2X^2\Gamma}\right), \text{ where } \Gamma = \sum_{i=1}^{n}\sum_{j=0}^{i}\left(\frac{\omega(i,j)}{k_{ij}}\right)^2.$$

Hence, for any positive real number c, our algorithm computes in $\mathrm{O}(\sum_{i=0}^{n}\sum_{j=0}^{i} k_{i,j})$ time a value Ψ satisfying $|\Psi - U| \le cX\sqrt{\Gamma}$ with probability at least $1 - 2e^{-c^2/2}$. To minimize the error bound $cX\sqrt{\Gamma}$ while keeping the time complexity $\mathrm{O}(kn^2)$, we define the number of buckets at node (i,j) by

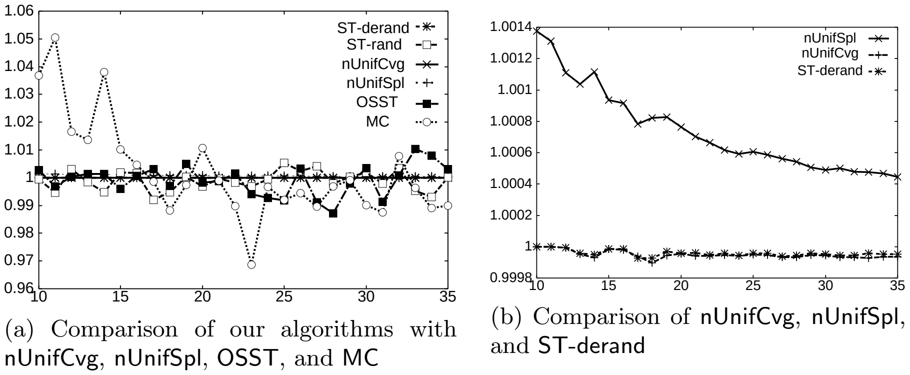
$$k_{ij} = \left\lceil \frac{k(n+1)(n+2)}{2} \times \frac{\omega(i,j)}{\sum_{i'=0}^{n}\sum_{j'=0}^{i'}\omega(i',j')} \right\rceil = \left\lceil \frac{k(n+2)}{2}\omega(i,j) \right\rceil.$$

Since $\Gamma \le 2/k^2$ and $\sum_{i=0}^{n}\sum_{j=0}^{i} k_{ij} \le (k+1)(n+1)(n+2)/2$, we have the following theorem, showing that the probabilistic error bound is $\mathrm{O}(X/k)$.

Theorem 5. *Let k be any positive integer and c be any positive real number. Then, our algorithm computes in $\mathrm{O}(kn^2)$ time a value Ψ satisfying $|\Psi - U| \le \sqrt{2}cX/k$ with probability at least $1 - 2e^{-c^2/2}$.*

3.4 Derandomization

Although the error bound $\mathrm{O}(X/k)$ of our algorithm shown in the last section is better than the previous approximation algorithms, our algorithm is randomized and therefore the error bound only holds with "high" probability. Hence, it is



(a) Comparison of our algorithms with nUnifCvg, nUnifSpl, OSST, and MC

(b) Comparison of nUnifCvg, nUnifSpl, and ST-derand

Fig. 3. Relative errors of approximate option prices computed by several algorithms

desired to derandomize our algorithm without losing its accuracy. One idea for derandomization is to take the weighted mean of running totals of the states in each bucket $B_i(j,h)$ as the value $R_i(j,h)$, as in the algorithm nUnifCvg by Dai et al. [9]. Although we have not yet proved the theoretical error bound $\mathrm{O}(X/k)$ for this derandomized version, it is experimentally shown that its error bound is better than the original one (see Sect. 3.5).

3.5 Experimental Results

We show some experimental results to illustrate the performance of our randomized approximation algorithm and its derandomized version. In particular, we compare the quality of the option price computed by our algorithms with those by other approximation algorithms. We implemented the full-path method to compute the exact price, and approximation algorithms such as the naive Monte Carlo method (MC), the AMO algorithms (AMO-LB, AMO-UB), the DHL algorithms [9] (nUnifDown, nUnifCvg, nUnifUp, nUnifSpl), and the OSST algorithm [13] (OSST). We denote our randomized and derandomized algorithms by ST-rand and ST-derand, respectively. The experiment is done by a Pentium IV 2.60CGHz PC and all programs are implemented in C++.

In the experiment, we consider a uniform model with $S_0 = X = 100$, $u = 1.1$, $d = 1/u$, $pu + (1-p)d = (1.06)^{1/n}$. The parameter k is set to 100 in the approximation algorithms except for MC. Recall that the positive integer k denotes the number of buckets used at each node for AMO-LB/UB and OSST while k is the average number of buckets used at each node for the DHL algorithms and ours. The Monte Carlo method MC takes $400n$ sample paths so that it runs in almost the same time as other approximation algorithms. In the experiment, only one trial is made for each algorithm.

Fig. 3 gives the result of the experiment in the range $n \in [10, 35]$, showing the ratio of the approximate prices computed by approximation algorithms to the exact price. The running time of the approximation algorithms are almost the same and less than 0.05 seconds, and the full-path method takes more than

9 hours when $n = 35$. The results of AMO-LB/UB and nUnifDown/Up are not shown in the graphs since the relative errors of these are always more than 0.2 and much worse than the relative errors of the other algorithms.

The graph (a) shows that the relative errors of nUnifCvg, nUnifSpl, and ST-derand are better than those of the other algorithms. In particular, our derandomized algorithm ST-derand performs much better than ST-rand. In the graph (b) we compare the three algorithms nUnifCvg, nUnifSpl, and ST-derand. We see that the relative error of ST-derand is quite accurate and as good as nUnifCvg. This result shows that the error bound of our derandomized algorithm ST-derand is much better than the error bound $\mathrm{O}(X/k)$ of the randomized algorithm ST-rand. It is an interesting open question whether ST-derand also has the theoretical error bound $\mathrm{O}(X/k)$, which is left for further research.

References

1. D. Aingworth, R. Motwani, and J. D. Oldham, Accurate approximations for Asian options, *Proc. 11th ACM-SIAM Symposium on Discrete Algorithms* (2000), 891–900.
2. K. Akcoglu, M.-Y. Kao, and S. V. Raghavan, Fast pricing of European Asian options with provable accuracy: single-stock and basket options, *Proc. Annual European Symposium on Algorithms 2001, Lecture Notes in Computer Science* **2161** (2001), 404–415.
3. K. Azuma, Weighted sum of certain dependent random variables, *Tohoku Mathematical Journal* **19** (1967) 357–367.
4. F. Black and M. Scholes, The pricing of options and corporate liabilities, *Journal of Political Economy* **81** (1973), 637–654.
5. P. Chalasani, S. Jha, F. Egriboyun, and A. Varikooty, A refined binomial lattice for pricing American Asian options, *Review of Derivatives Research* **3** (1999), 85–105.
6. P. Chalasani, S. Jha, and I. Saias, Approximate option pricing, *Algorithmica* **25** (1999), 2–21.
7. P. Chalasani, S. Jha, and A. Varikooty, Accurate approximation for European Asian options, *Journal of Computational Finance* **1** (1998), 11–29.
8. J. C. Cox, S. A. Ross, and M. Rubenstein, Option pricing: a simplified approach, *Journal of Financial Economics* **7** (1979), 229–263.
9. T.-S. Dai, G.-S. Huang, and Y.-D. Lyuu, Extremely accurate and efficient tree algorithms for Asian options with range bounds, 2002 NTU International Conference on Finance, National Taiwan University, Taiwan, May 2002.
10. P. Glasserman, *Monte Carlo Method in Financial Engineering*, Springer, Berlin, 2004.
11. J. C. Hull, *Options, Futures, and Other Derivatives, Fifth Edition*, Prentice Hall, Upper Saddle River, NJ, 2002.
12. R. Motwani and P. Raghavan, *Randomized Algorithms*, Cambridge Univ. Press, London, 1995.
13. K. Ohta, K. Sadakane, A. Shioura, and T. Tokuyama, A fast, accurate and simple method for pricing European-Asian and Saving-Asian options, *Algorithmica*, to appear.