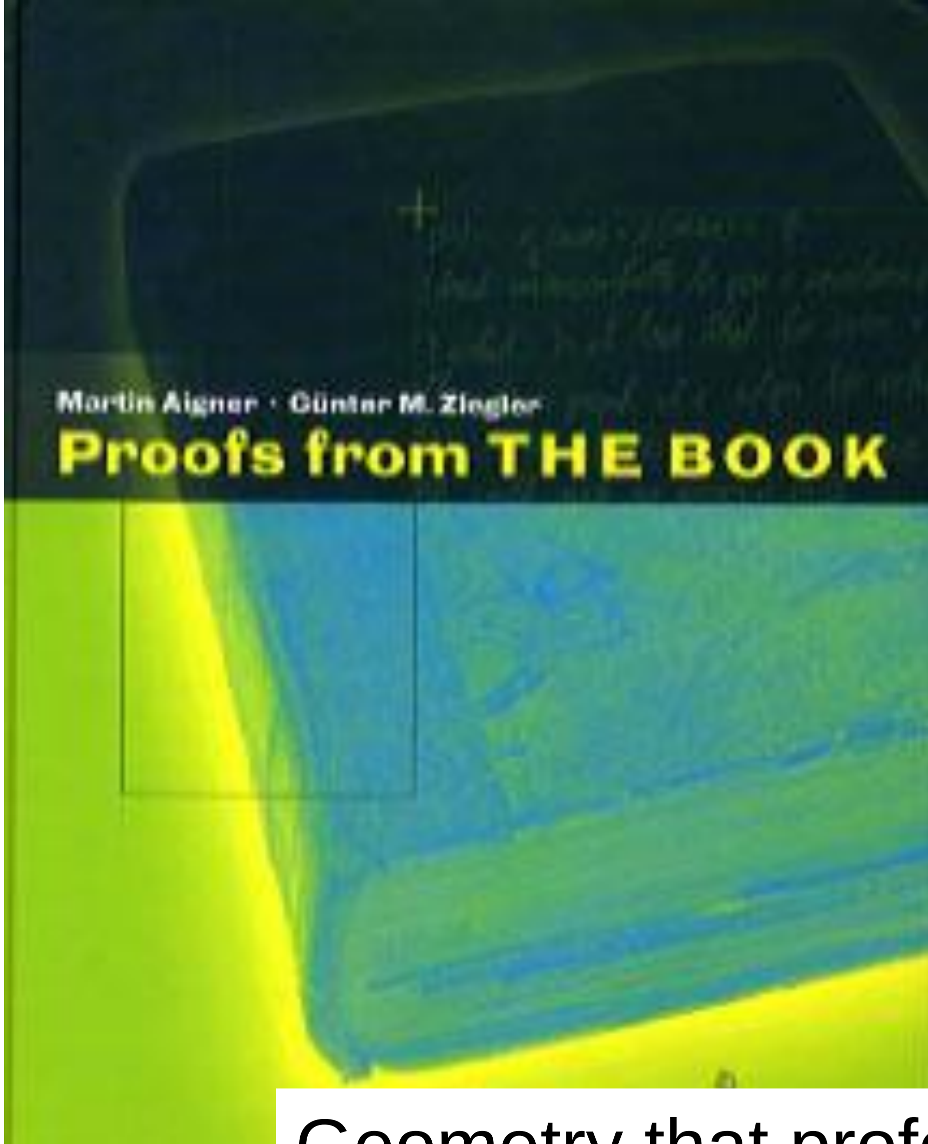

Mathematics that the professor loved

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東北大学

Geometry that professors love

博士たちの愛する幾何

Agenda

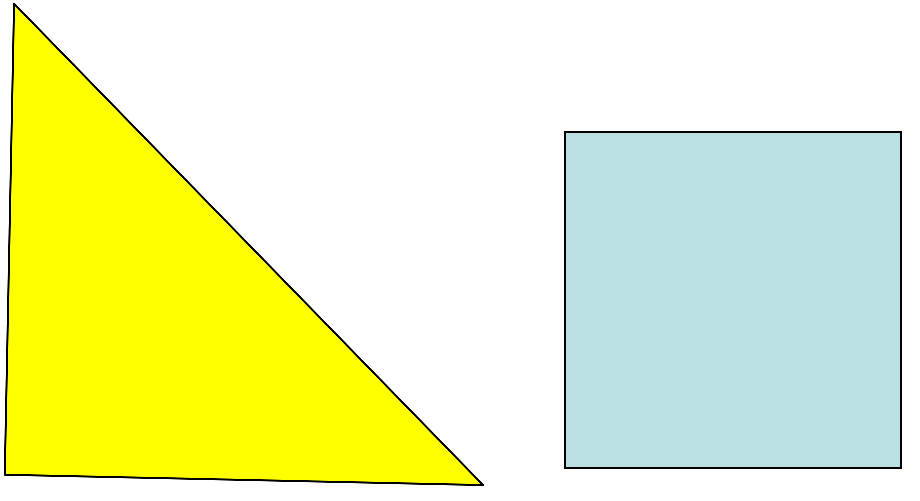
- Great problems in mathematics
- 23 problems proposed by Hilbert（1900）
    - Hilbertの23の問題(1900)
  - The list of grand challenges of modern mathematics
    - 現代数学をリードした数学プラン
- How great mathematicians think
  - Equi-decomposability problem (分割同値問題）
  - Theorem of Bolyai and Gerwien
  - What Gauss thought….
- Wide scope is powerful tool for a scientist
  - Geometric problem is often not solvable by only using geometry
    - Number theory and group theory to solve geometric problems
  - Same in any area of science and engineering

Equi-decomposability

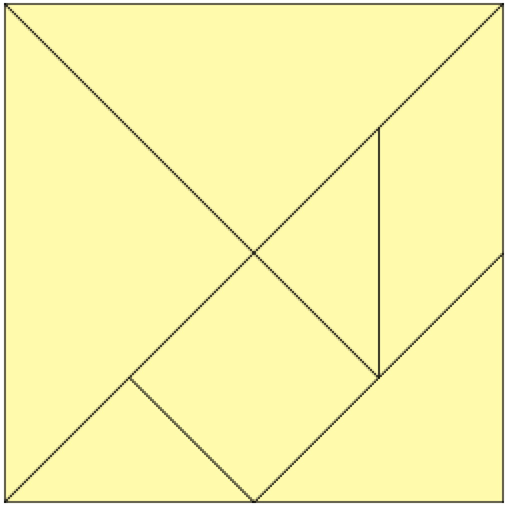
- Given two polygons P and P’ , can we show that they have the same area?
- 二つの多角形PとP’があるとき、それらが同一面積を持つことをどうやって示す？
- Equidecomposing polygons
  – Decompose P into a finite number of pieces, and assemble them to make P’
  – Pを部分多角形に分割して、組み立てなおしてP’を作れるか？

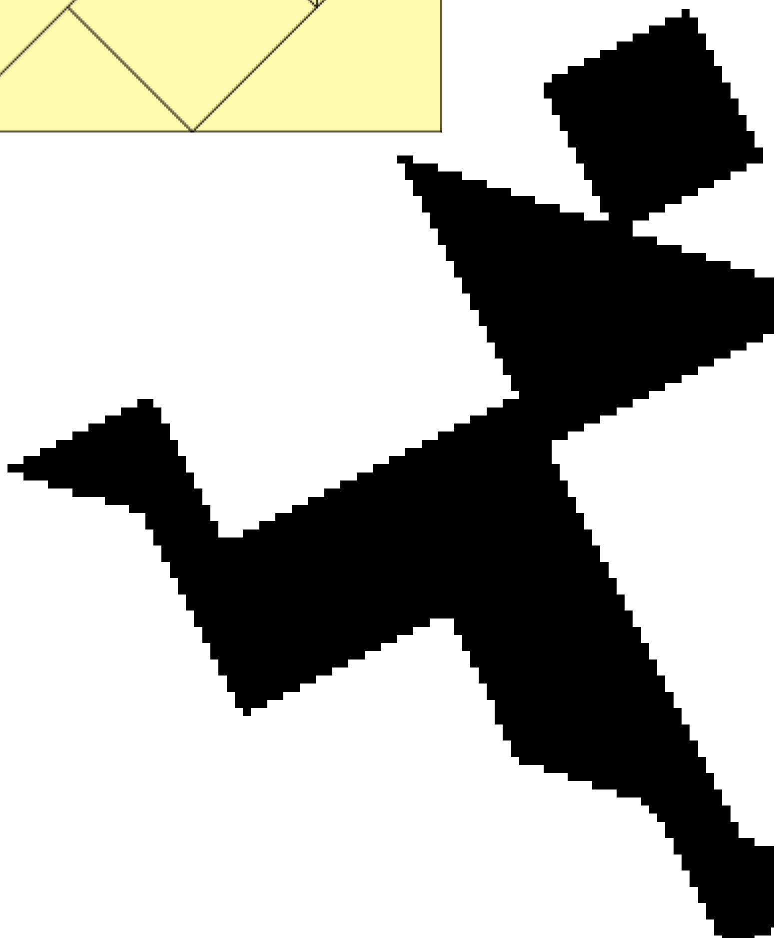
Theorem of Bolyai-Gerwien（１８３２）

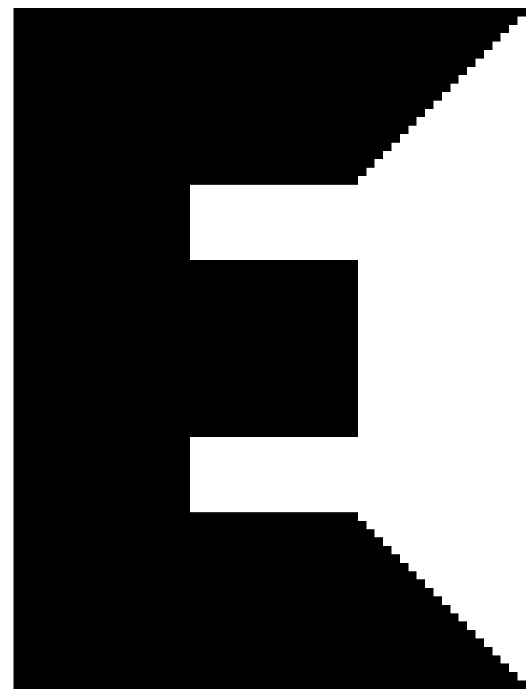
- Any pair of polygons with same area are Equidecomposable(分割同値）
- Example:

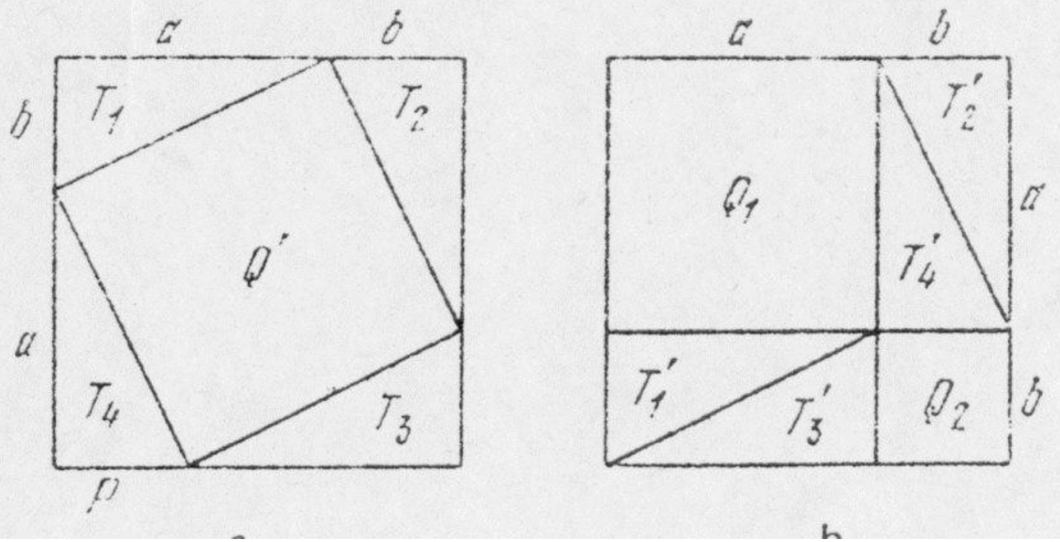


タングラムパズル





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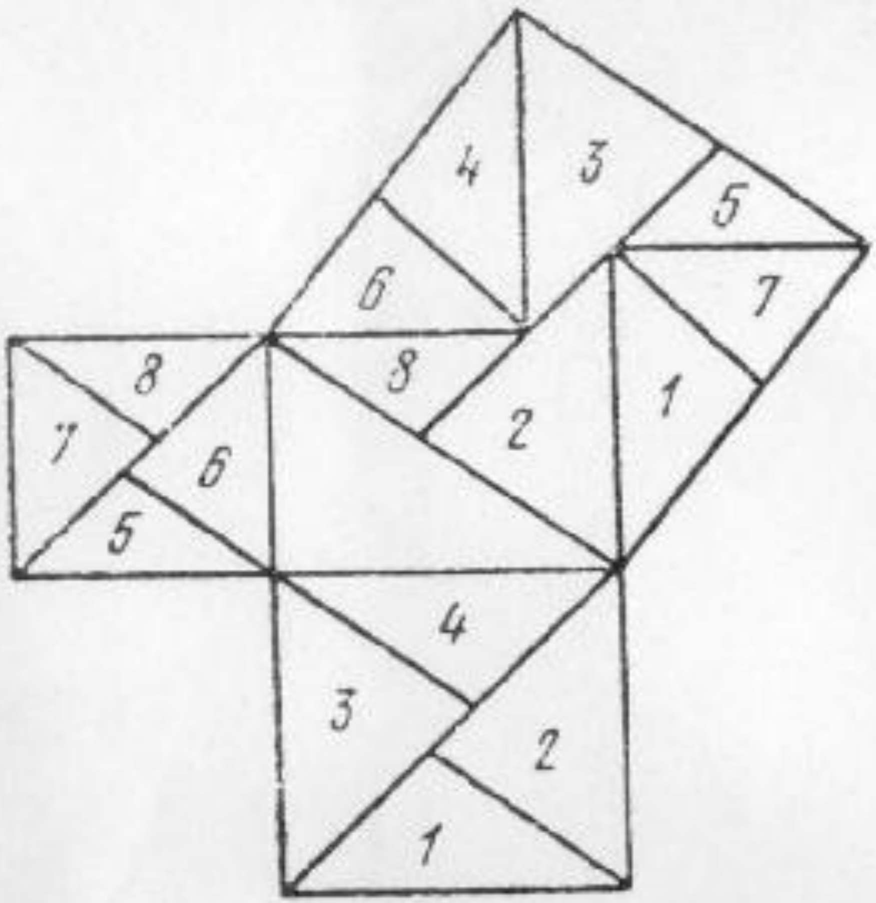
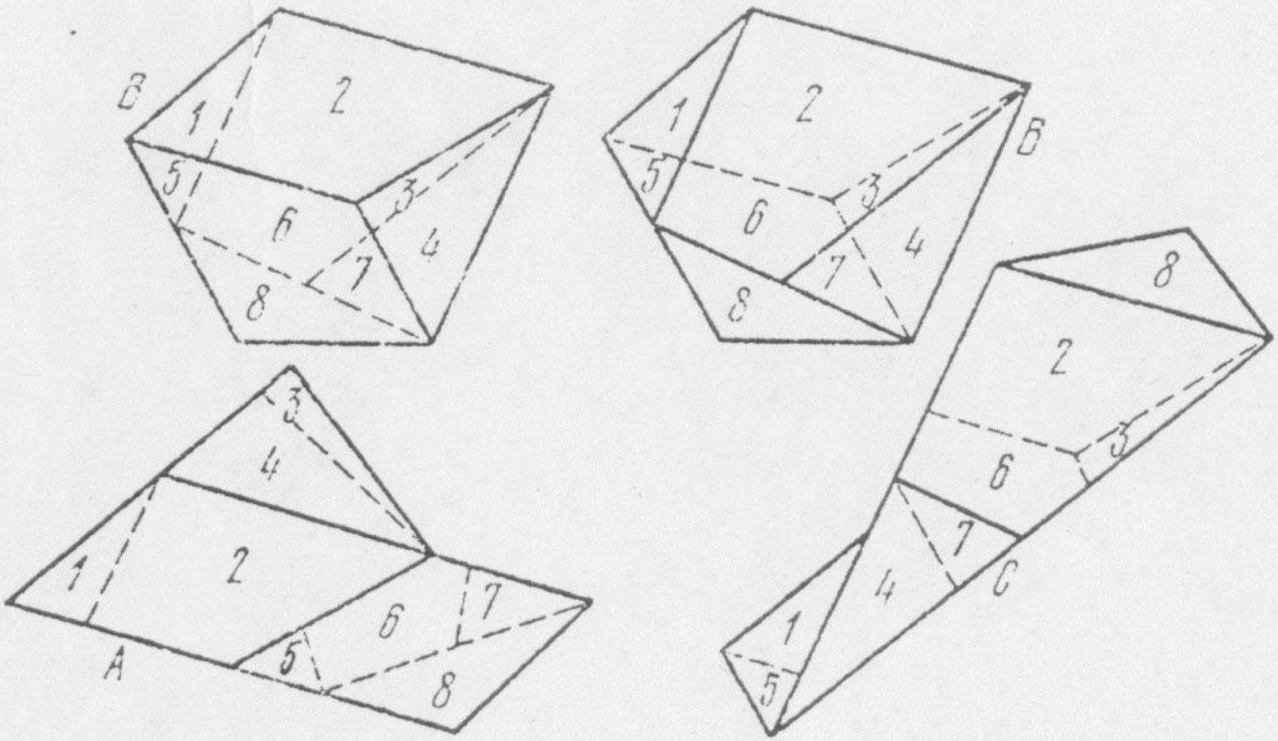


FIG. 20

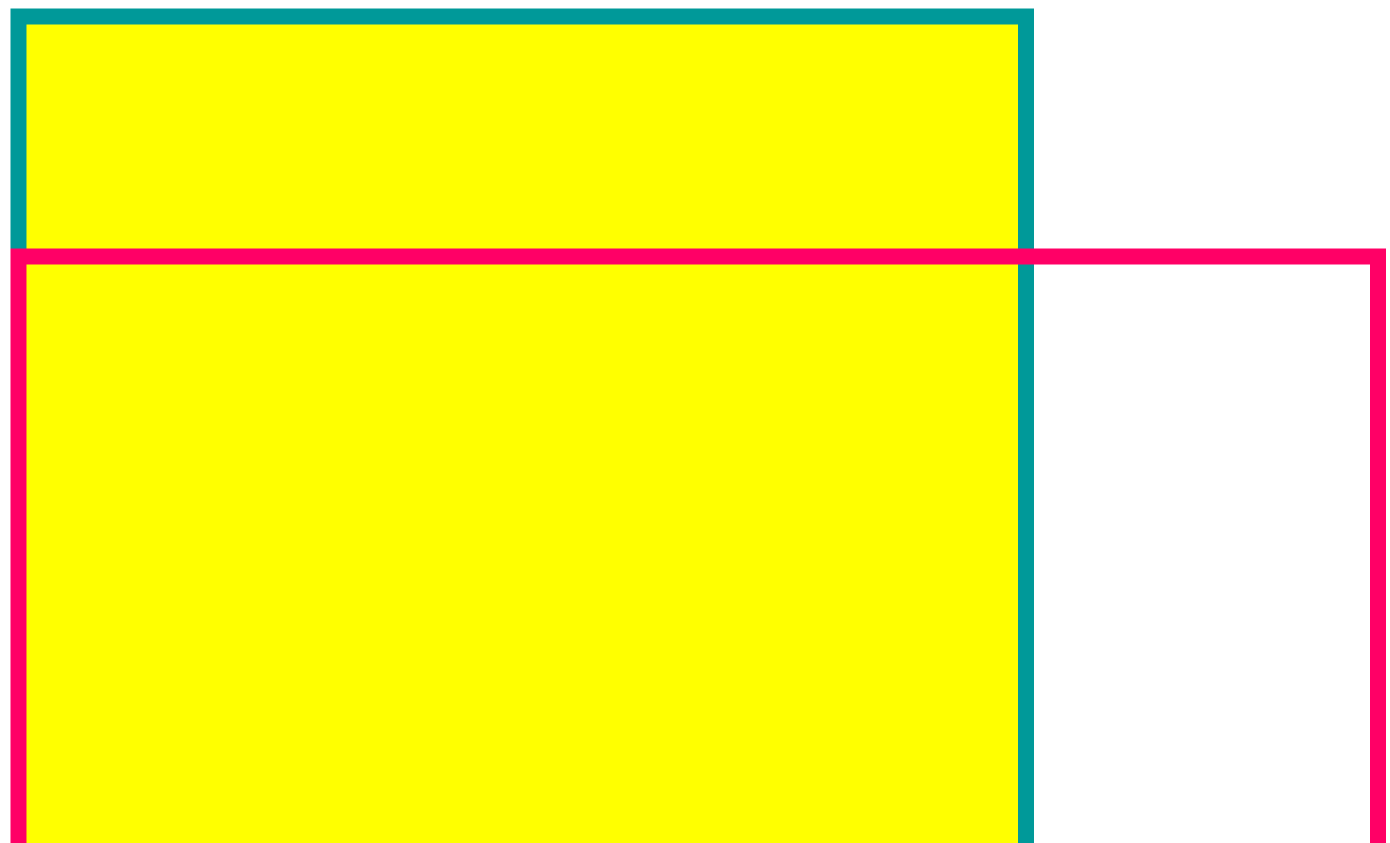


a b

FIG. 27

演習問題による証明
Solution by a series of exercises

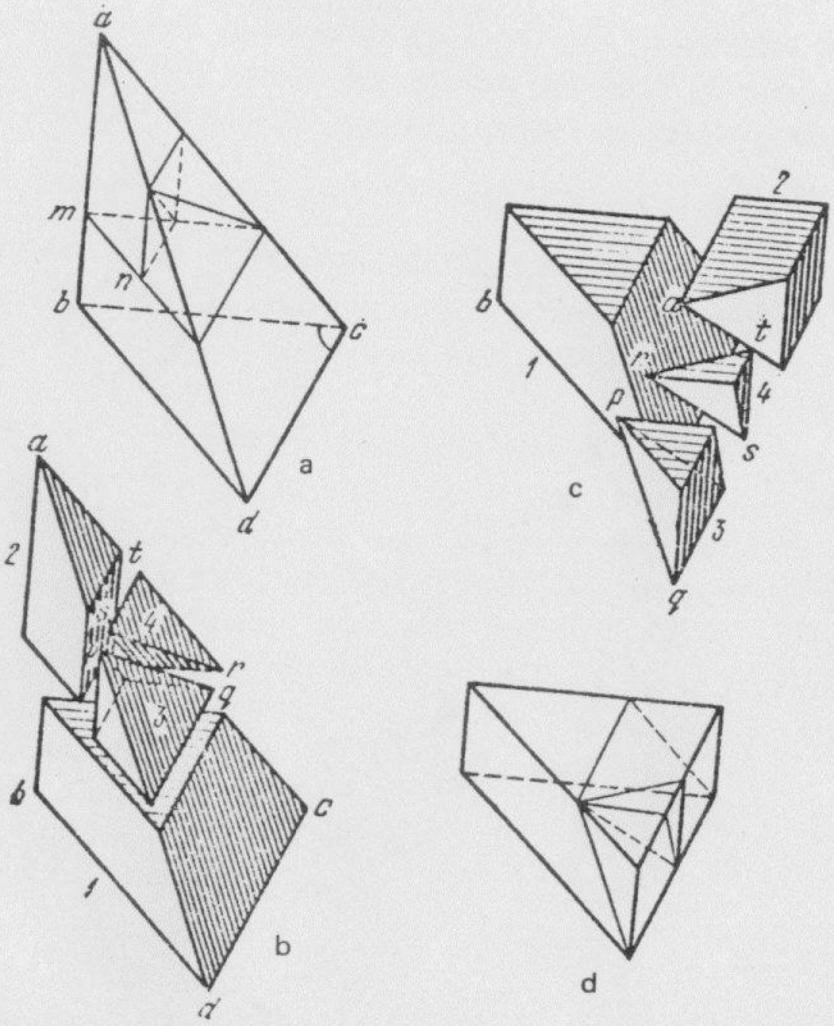
- Show that a unit area rectangle is equidecomposable to a unit square
  - If the side edge is rational, not difficult, but in general..
  - 面積1の長方形は単位正方形と分割同値であることを示せ
- Using the above fact, sbhow that any pair of equi-area polygons are equidecomposable.
- 上の事実を用いて、任意の2つの面積が等しい多角形が分割同値であることを示せ
- Theorem of Bolyai and Gerwien
  - Farkas Boyai: Invented non-Euclidean geometry with his son, Janos Bolyai



Equideomposablity means

- In 2dim, decomposition, scaling, translation, and rotation can create any "shape"
- What is area of a figure? It is the size of square equidecomposable to the figure.
- Gauss： Different in 3D space
  - 三角錐の体積の公式をどうやって小学生に教えるか？

    How to teach the volume formula of tetrahedra to kid?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The 3rd problem of Hilbert

- Can we show that two prisms with the same base area and height are equidecomposable?
  - Solved in 1900: Earliest solution among 23 problems
- Max Dehn (22 years old）:
  - Unit volume simplex is not equidecomposable to the unit cube
    - 正4面体と立方体は分割同値ではない
  - This is number theory problem

Dehn’s idea: 1

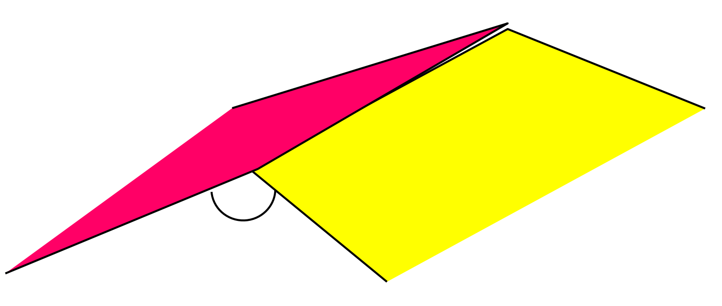
Two values for an edge e of a polytope P

多面体Pの辺e について、2つの量を考える。

１:   Length *l*(e)

2:   Angle θ(e) between two faces

３:   Use the pair（l(e), θ(e) mod 2π）

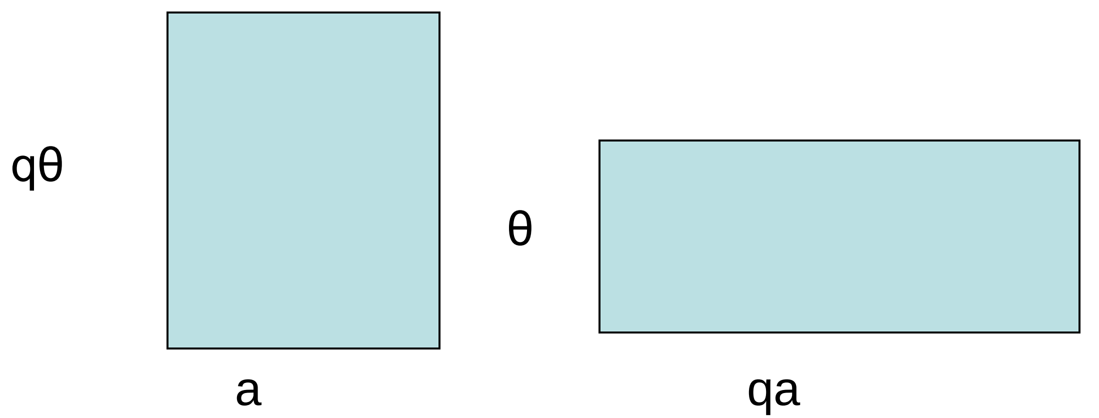


Idea of Dehn: 2

We define equivalence that for a rational number q $(a, q\theta) = (qa, \theta)$

2次元のイメージ図:　面積不変、分割も簡単

3次元の場合でも、辺の長さがc倍で角度1/cだと、辺の周りでの体積は不変に思える(これは動機なので、当面証明不要)



Tensor product over the rational field Q： Almost multiplication, but restricted (like filtration)

$$a \otimes q\theta = qa \otimes \theta \ \mathit{if} \ \ q \in \mathrm{Q}$$

$$a \otimes \theta + a \otimes \tau = a \otimes (\theta + \tau)$$

$$a \otimes \theta + b \otimes \theta = (a + b) \otimes \theta$$

$$0 \otimes \theta = a \otimes 0 = 0$$

注意： 物理や量子計算などでの通常のテンソル積では、
項 はベクトルで、Qの代わりに実数や複素数を用いる

Example：

$$\sqrt{2} \otimes (\pi / 3) = (\sqrt{2} / 6) \otimes 2\pi = (\sqrt{2} / 6) \otimes 0 = 0$$

$$1 \otimes \sqrt{2}\pi = 3 \otimes (\sqrt{2}\pi / 3) \neq 0$$

$$a \otimes \theta + a \otimes (\pi - \theta) = a \otimes \pi = 0$$

Dehn invariant

$$f(P) = \sum_{e:edges} l(e) \otimes \theta(e)$$

Lemma： If we decompose P into two polytopes P1 and P2

f(P)= f（P1)+f(P2)

Theorem： If P and P’are equidecomposable, f（P)=f(P‘)

つまり、Dehn不変量が異なる多面体は互いに分割同値でない

Dehn invariants of cube and simplex

- Dehn invariant of any cube is 0
  - Why?
- The Dehn invariant of a simplex of unit-edge length is:

$$f(P) = \sum_{e:edges} l(e) \otimes \theta(e) = 6 \otimes \theta(e)$$

$$= 6 \otimes \arccos(1/3)$$

There is not quotient number q such that cos(qπ)=1/3

There is no quotient number q such that $\cos(q\pi)=1/n$ for odd n

$\cos\ (k+1)\, x = 2 \cos x \cos kx - \cos\ (k-1)\, x$

Thus, if $\cos x = 1/n$, $\cos k\, x = A(k)/\, n^k$
where $A(k)$ is coprime to n.

Thus, cos kx cannot be an integer for any k.

But, if $q = a/b$, $\cos 2bx= 1$. A contradiction.

The implication of Dehn’s theorem

- Difference between 2-dim and 3-dim geometry
  - Volume is not really analogue of area
- We need analytic method to show the volume formula of a tetrahedra.
- Invariant is fundamental for classification of geometric objects
  - Area, volume
  - We need more
- Algebraic method is fundamental for geometry
  - Dehn’s surgery, Dehn’s twist
  - Homology (Poincare’s claim in 1900)

More about it

- If two polytopes P and Q have same volume and Dehn invariant, can we decide whether P and Q are equidecomposable?
- Difficult looking：Isomorphism problem of graphs, geometric objects, etc
- After 65 years、Sydler solved it by using group theory,
  - Yes, P and Q are always equidecomposable.

- Algorithmic problems
  - Computation of Dehn invariant
  - Finding exact method for equi-decomposing
  - Minimum decomposition for 2-dim problem
    - Probably NP-hard, how to approximate??

Modern geometry

- Invariance and transformation
  - Invariants and classification: Homology, Homotopy, etc
    - Poincare's claim: Homology classify geometric surfaces
      - Wrong!: "Exiotic sphere" exists.
    - Poincare's conjecture(1904): Homotopy determines spheres.
      - Perelman（2003) proved it
    - Thurston program, Morse theory
  - Graph isomorphism
    - Invariants： Order sequence、Neighbor degree set
    - We have not yet found good invariants.
- Invariants and information science
  - Geometric matching and geometric information retrieval
    - E.g., We can obtain geometric objects in a database that has the same homology as a torus (donut)
  - We need invariants to represent more detail shape.
    - This is not completely solved, and very challenging problem
- Serious: We do not know whether our space is exotic or not!
- Dehn seminar: organized by Prof. Fujiwara