Design & Analysis of Information Systems

Mathematics in Computer Science.

This year's topic is

Computational Geometry.

Mathematics in computer science

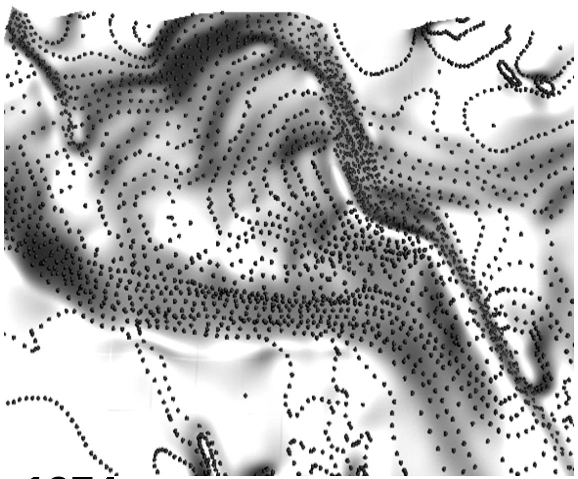
- Mathematics seeks for elegant solutions
  - "Solution" is not just "showing the answer".
  - It is important to describe the solution process.
- Computer science seeks for elegant and efficient algorithms
  - Algorithm: Concrete description of process.
  - Algorithms lead to our modern life.
- Mathematics is vital in algorithm design
  - Solve seemingly-impossible tasks.

Computational Geometry

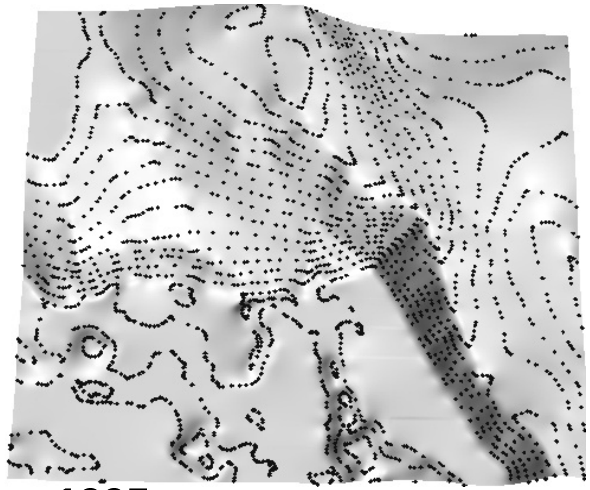
- Design algorithms for geometric problems
  - Many modern applications
    - GIS, Graphics, Geometric Modeling, Robotics, Multidimensional Database, Computer Vision.
  - Fast processing of massive data
    - Giga pixel data = 1000,000,000 data in a single digital picture
- Elegant geometry for algorithm design
  - Discrete geometry, etc.
  - Exciting intellectual puzzles (知的パズル）

Diverse elevation point data: density, distribution, accuracy

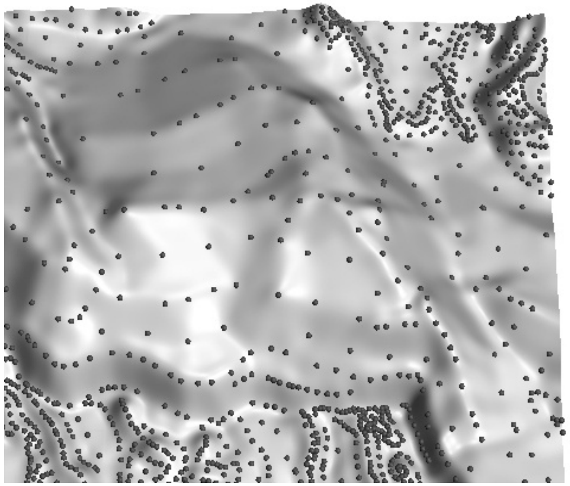
Photogrammetry 0.76m v. accuracy (5ft contours)



1974



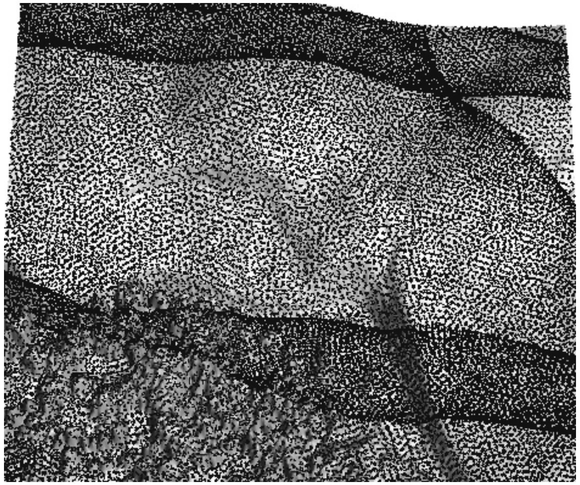
1995



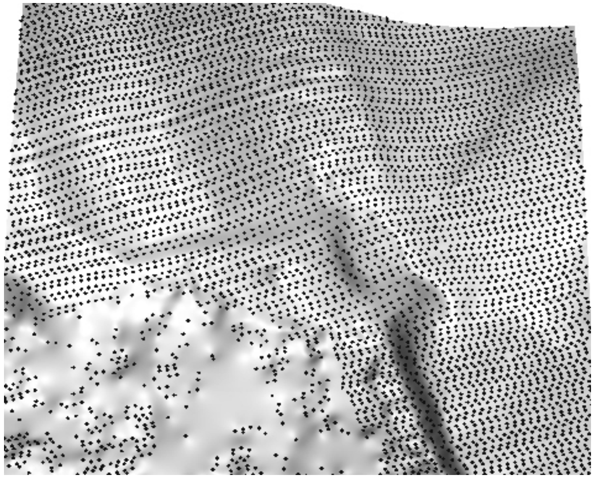
1998

Lidar 0.15m v. accuracy; altitude 700m and 2300m

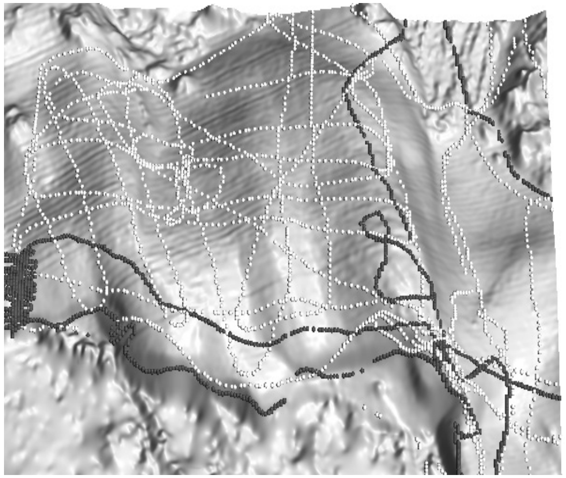
RTK-GPS 0.10m v. accuracy



1999



2001



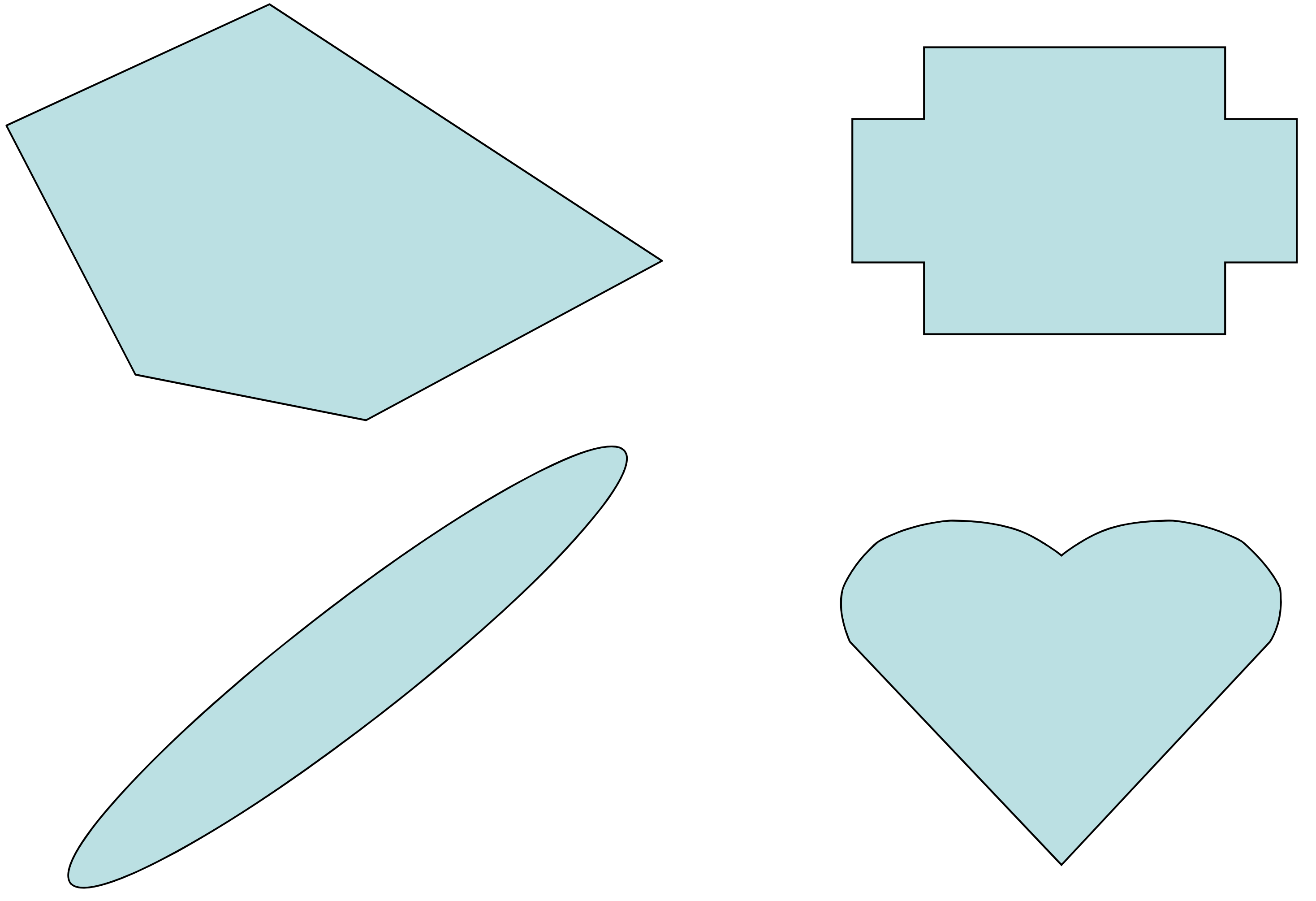
2004

100 meters

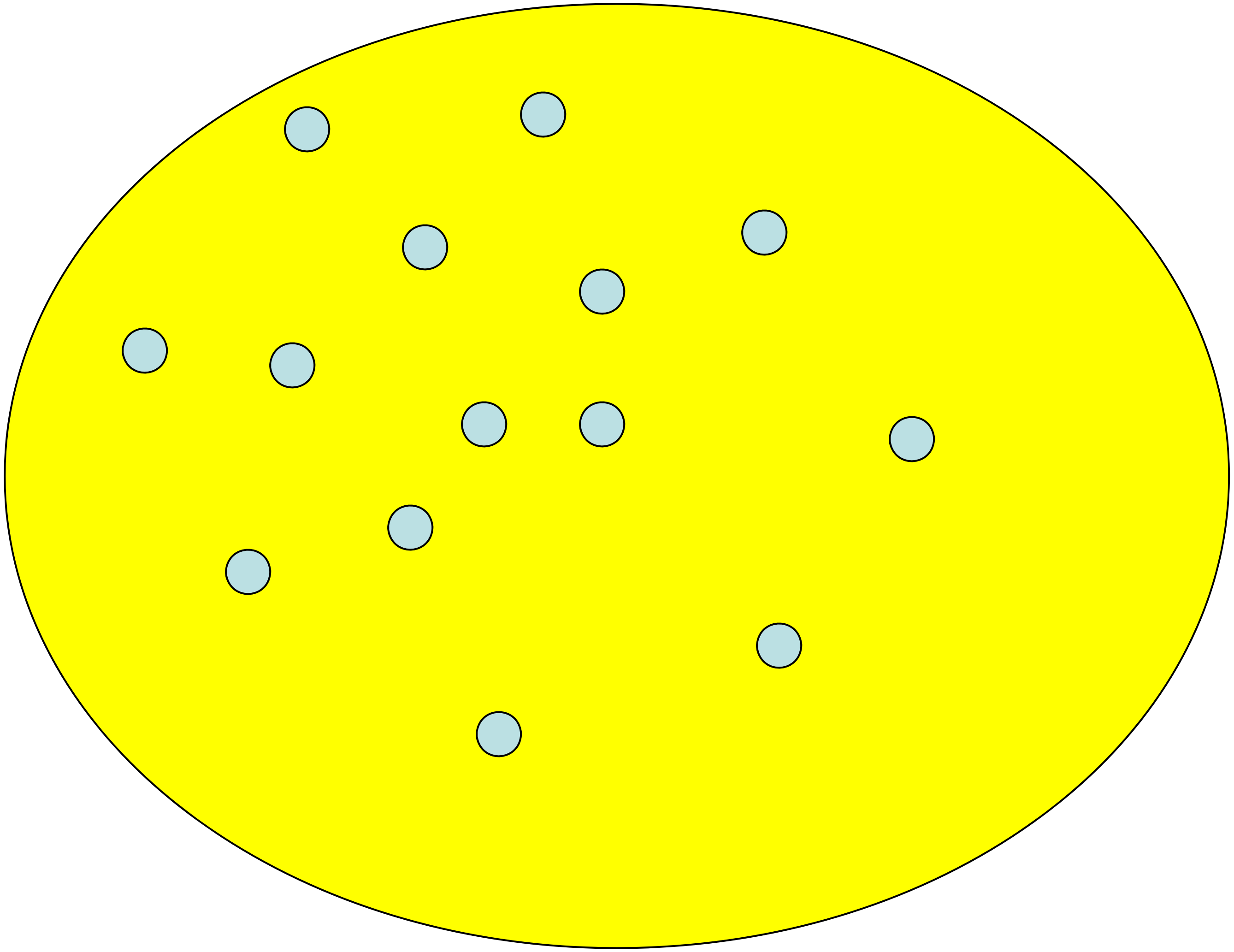
Example of geometric computation

- Convex hull computation（凸包の計算）
  – A showcase of algorithmic techniques
- Given a set S of n points in a plane, compute its convex hull
  – Convex set: A set X such that for any two points p and q in X, the segment pq is in X
  – Convex hull CH(S) of S : Minimum convex set containing S
    - Question: Is convex hull well-defined (i.e., always uniquely exists) ?

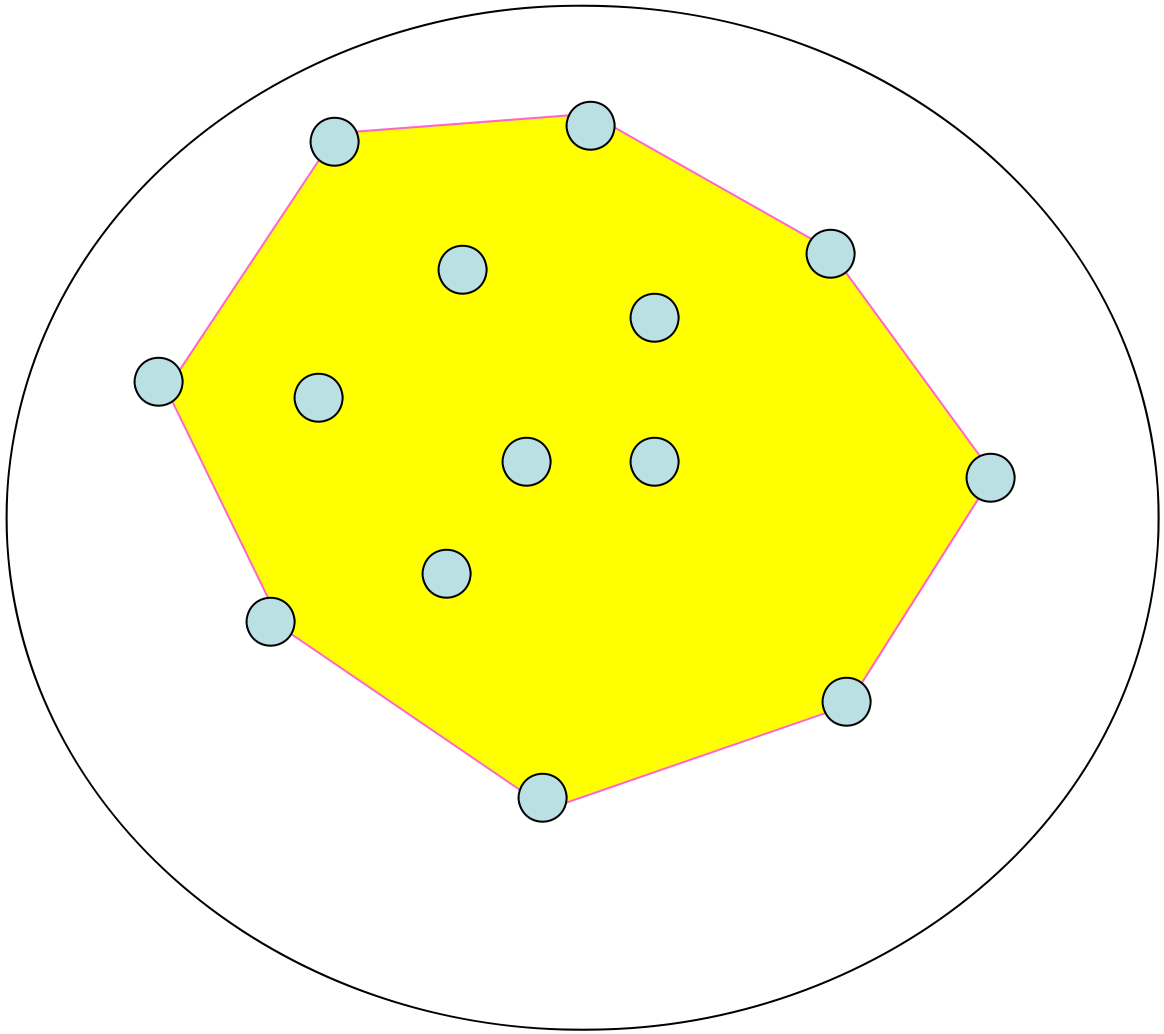
Convex Sets ?



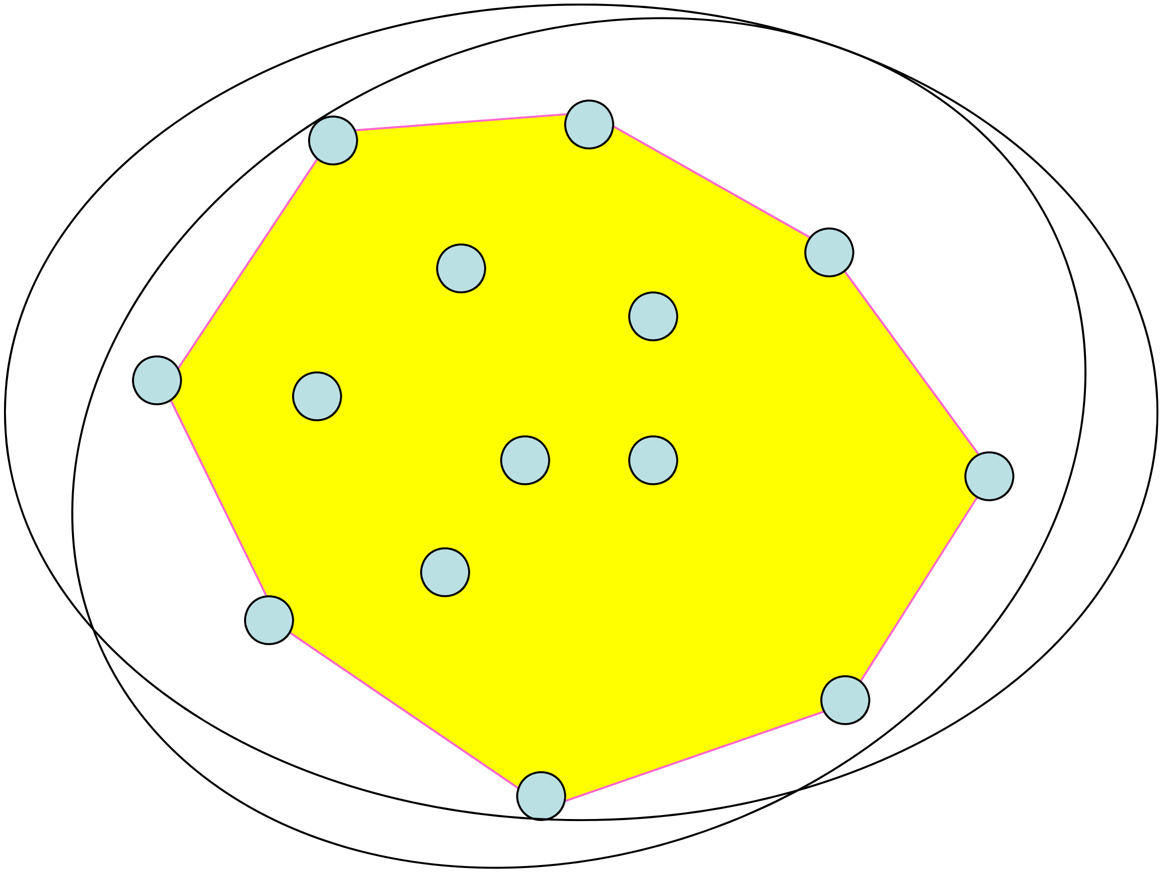
Convex set containing S



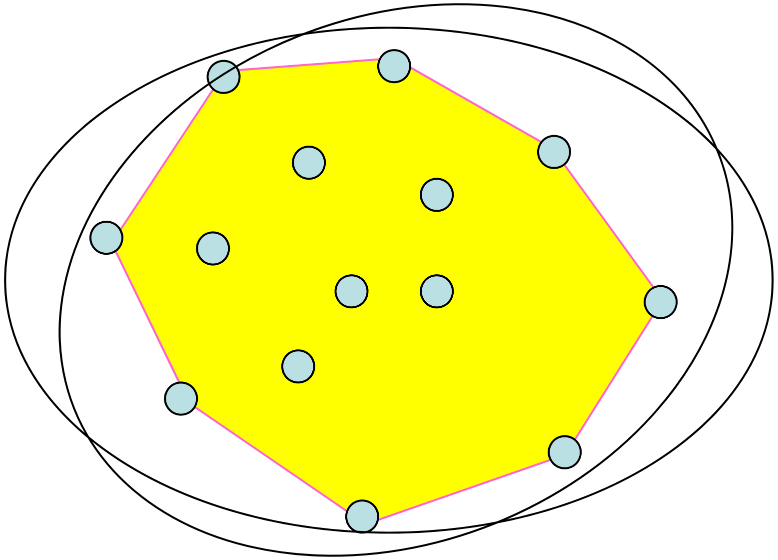
Convex hull of S



Every convex set containing S must also contain CH(S)



Convex hull exists



$$CH(S) = \bigcap_{X:convex,\ X \supset S} X$$

- Nice mathematical representation.
- But it is hard to use in computation.

Convex Hull Computation

- Given a set S of n points, compute CH(S)
- Can you design an algorithm?

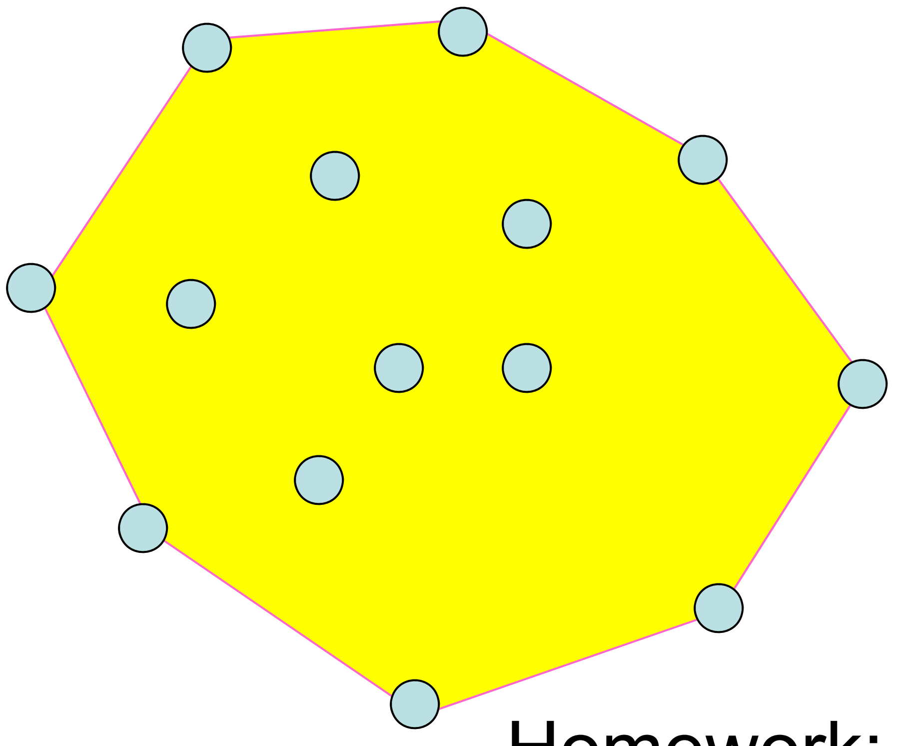
$$CH(S) = \bigcap_{X:convex, X \supset S} X$$

Very nuisance formula. The right hand side is intersection of infinite number of convex sets

We should transform this "cold" formula into more "friendly" one.

Characterization.

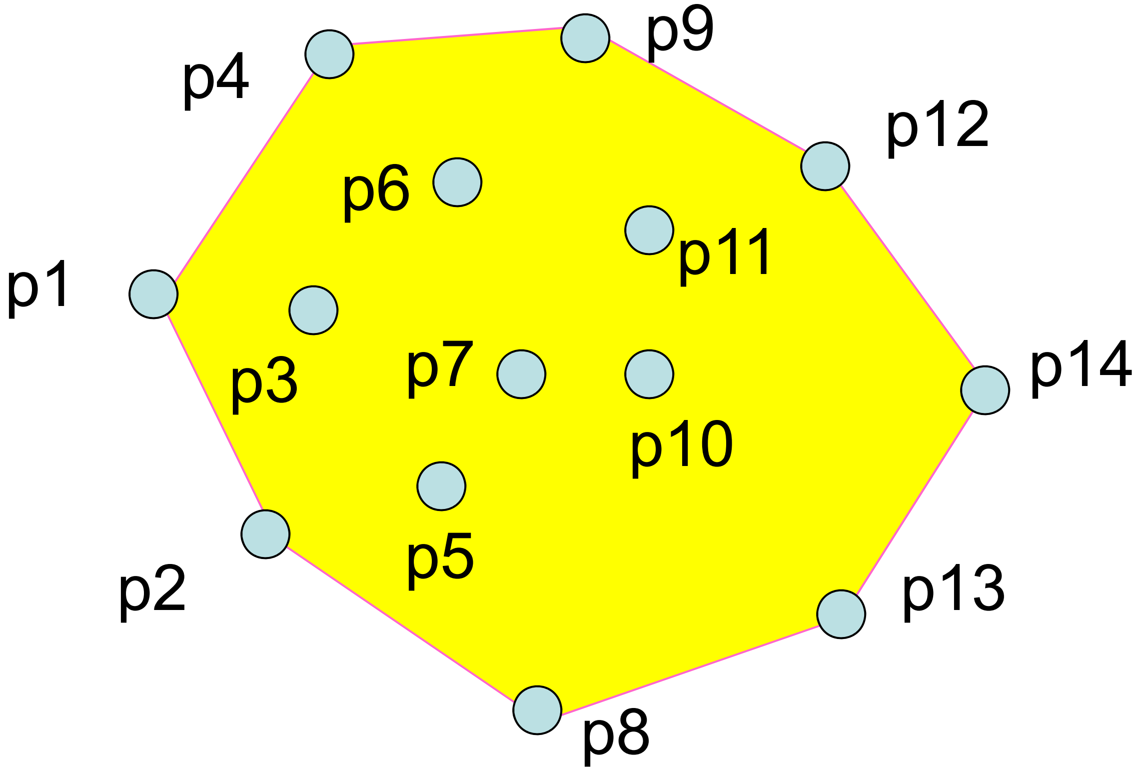
1. CH(S) is a convex polygon
2. Vertices of CH(S) are points of S
3. CH(S) contains all points of S



Homework: Prove it!

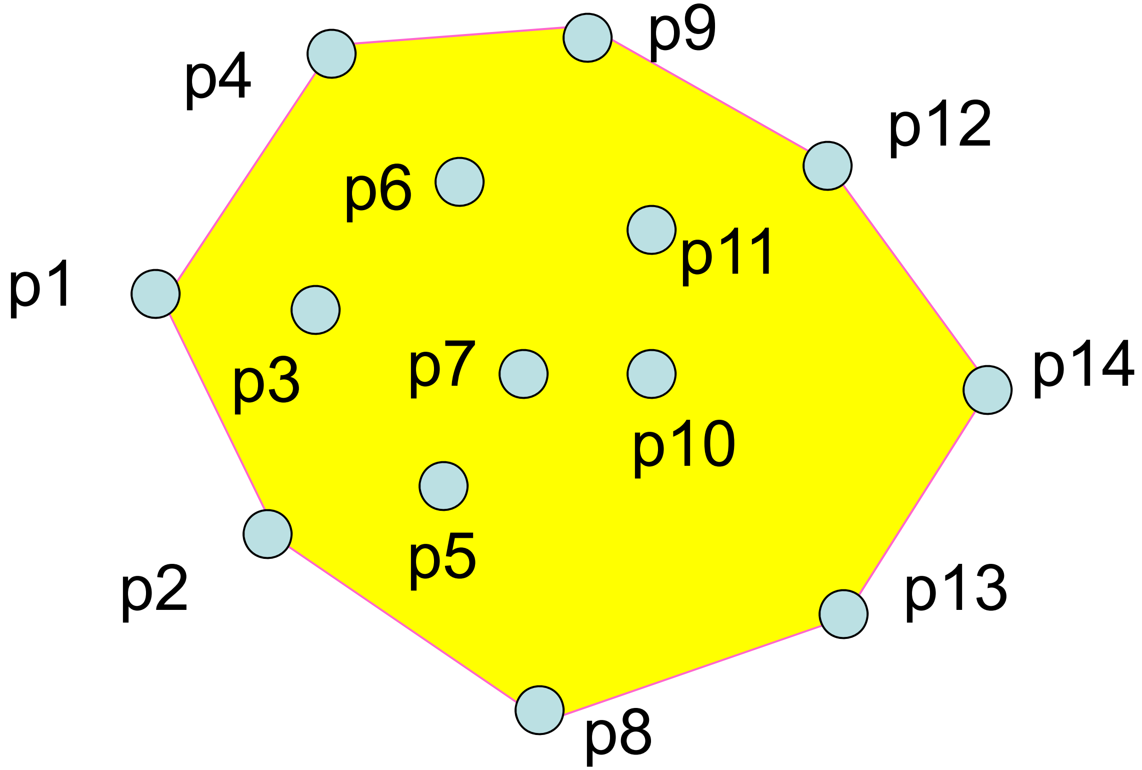
A representation of CH(S):

the list of vertices in a clockwise order starting from the leftmost one



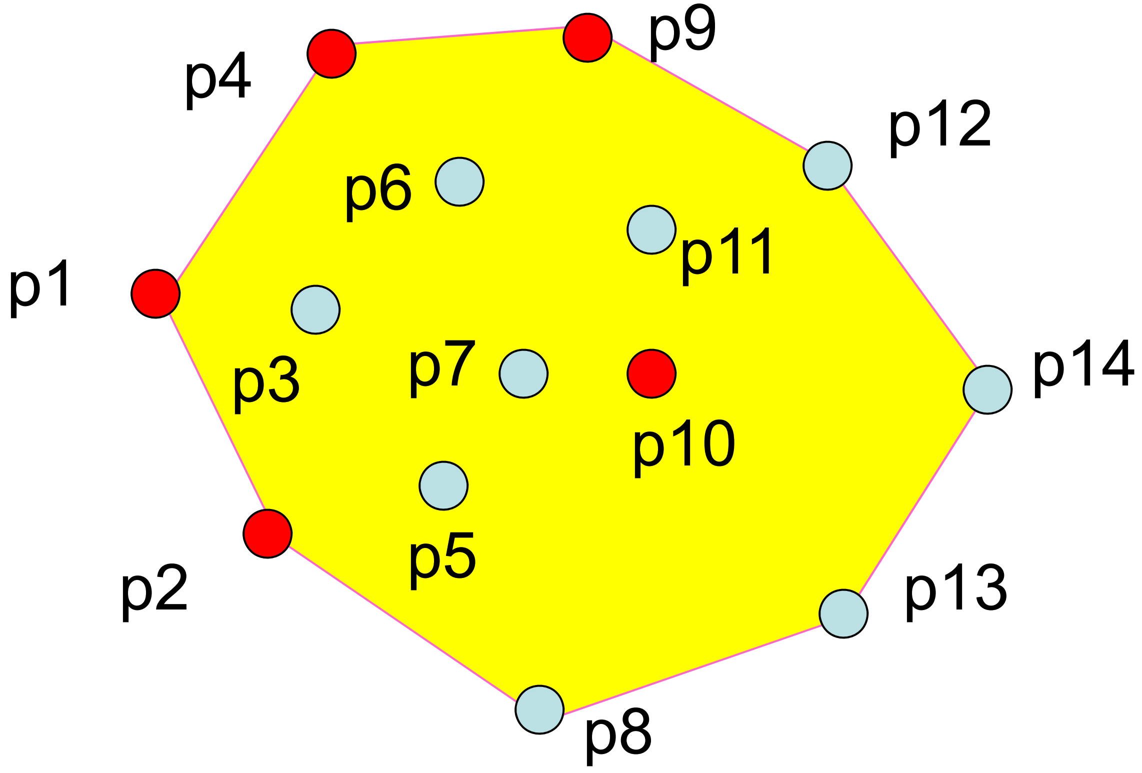
Question: Show CH(S) of the above picture in the above representation.

Now, the problem is in the discrete and finite world



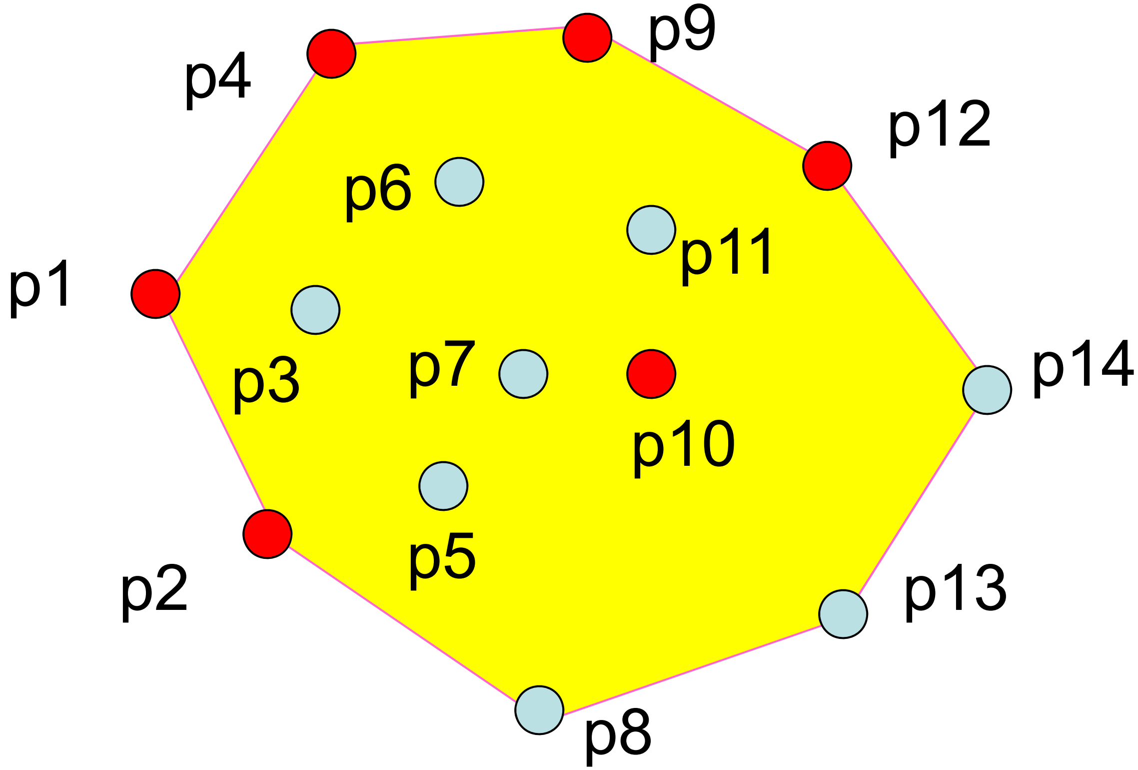
Find the (partial) permutation of S forming the convex hull.

Verification problem



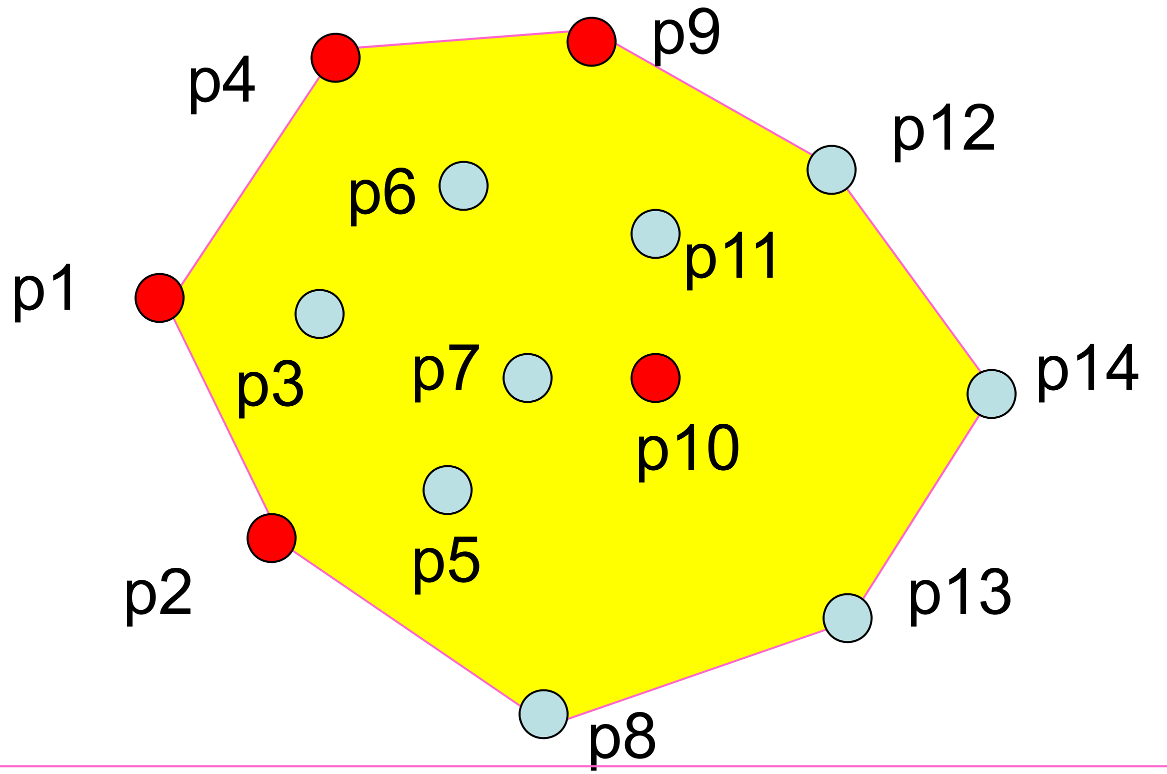
Given a list (p1, p2,p10,p12,p9,p4), verify whether it is CH(S).

Verification problem 1



Given a list (p1, p2,p10,p12,p9,p4), verify whether it gives a convex polygon

Verification problem 2



If the list (p1, p2,p10,p12,p9,p4) gives a convex polygon, show all other points are contained in it.

A brute-force algorithm

- Algorithm 1:
  - Generate all possible partial permutations of S
  - For each permutation P, verify it gives CH(S)
- Questions
  - Is the above algorithm always correct?
  - How much time does it take if n= 1000.
- Conclusion: We need a better algorithm.
- Question: Please consider a better algorithm than this!

Analysis of Algorithm

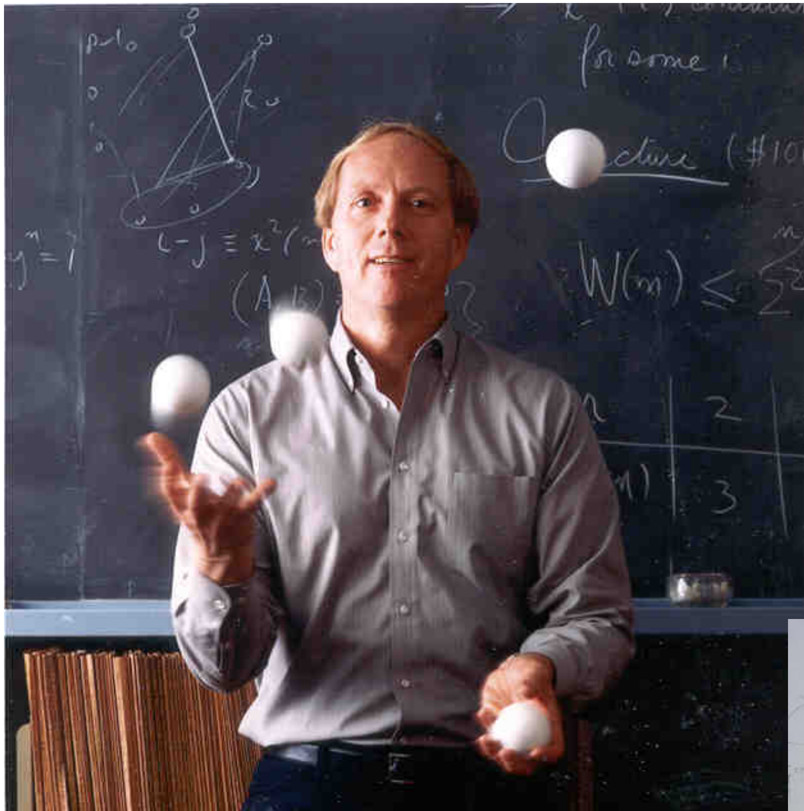
- Time Complexity
  - Given an input of size n (words/bits), how many basic steps are required in an algorithm?
    - Arithmetic operations
    - Comparisons, Data Access (read, write)
    - Floor/Ceiling [314.1592] = 314
  - T(n): number of basic steps
  - Asymptotic time complexity
    - $T(n) < c\, f(n)$ for a suitable constant c and a familiar function f(n)
    - We write $T(n) = O(f(n))$
- Classification of time complexity
  - Polynomial time algorithm: f(n) is a polynomial in n
    - Linear time algorithm: $T(n) = O(n)$
    - Quadratic time algorthm: $T(n) = O(n^2)$
  - Exponential time algorithm : e.g., $f(n) = 2^n$
  - Unbounded time algorithm : No such f(n)

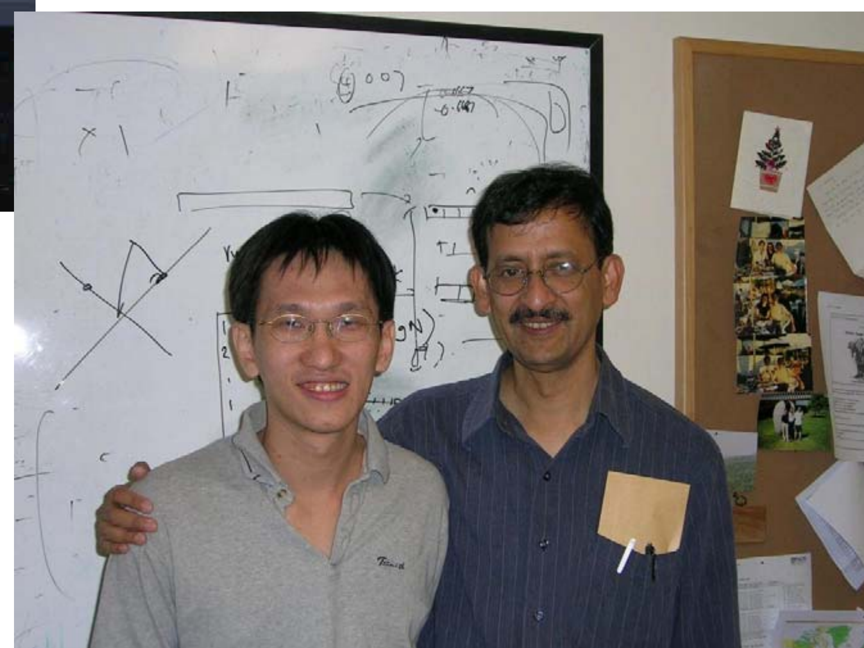
Complexity of a problem

- Complexity of a problem X
  - The complexity of X is $O(f(n))$ if there is an algorithm to solve X in $O(f(n))$ time
  - The complexity of X is $\Omega(f(n))$ if there is no algorithm to solve X in $o(f(n))$ time
    - $o(f(n))$: strictly smaller than $O(f(n))$
  - The complexity of X is $\Theta(f(n))$ if it is both $O(f(n))$ and $\Omega(f(n))$
- Complexity class of a problem
  - A problem X is in class P if there is a polynomial time algorithm to solve X, that is, the complexity of X = $O(f(n))$ for a polynomial $f(n)$.

Typical problems in class **P**

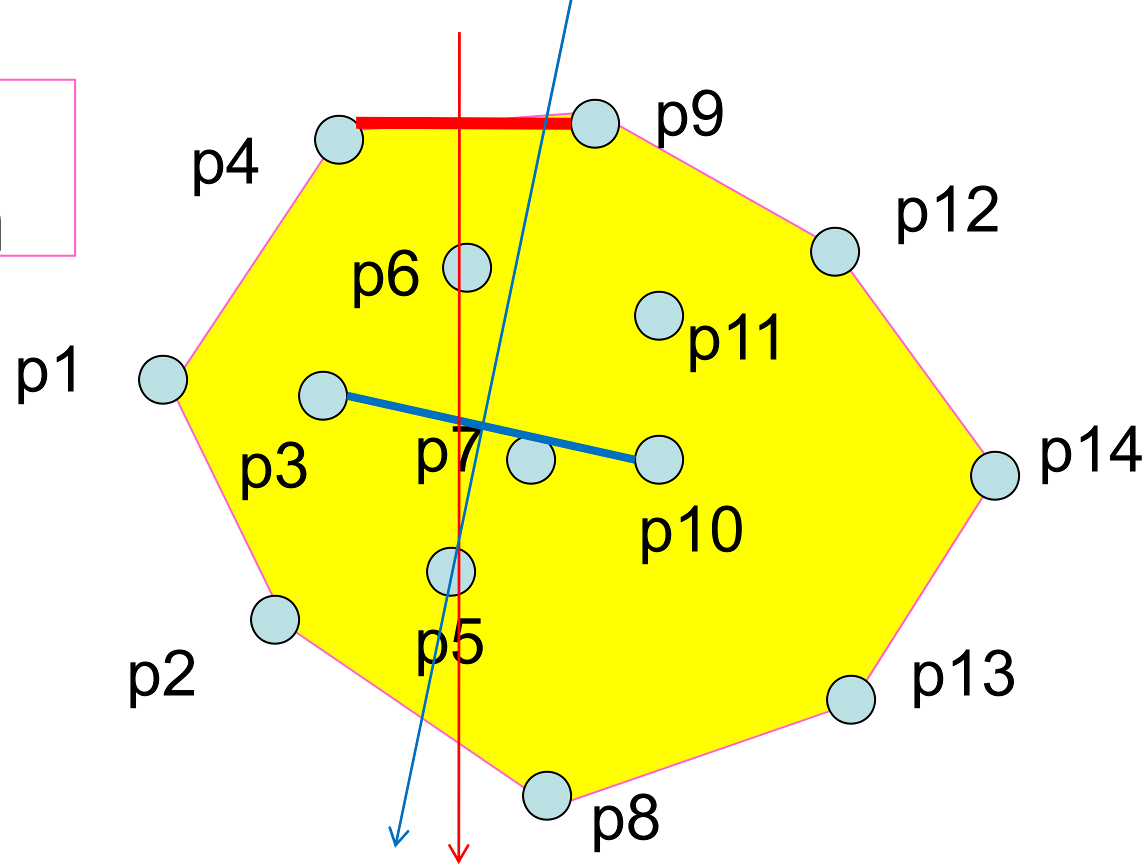
- Finding maximum element in a given set of n numbers : $\Theta(n)$ time
- Sorting n numbers: $O(n \log n)$ time
  - $\Theta(n \log n)$ if we restrict the computation model
- Computing the "distance" of two DNA sequences of length n: $O(n^2)$ time
- Computing convex hull of n points in the plane
  - How to do it? What is the time complexity?





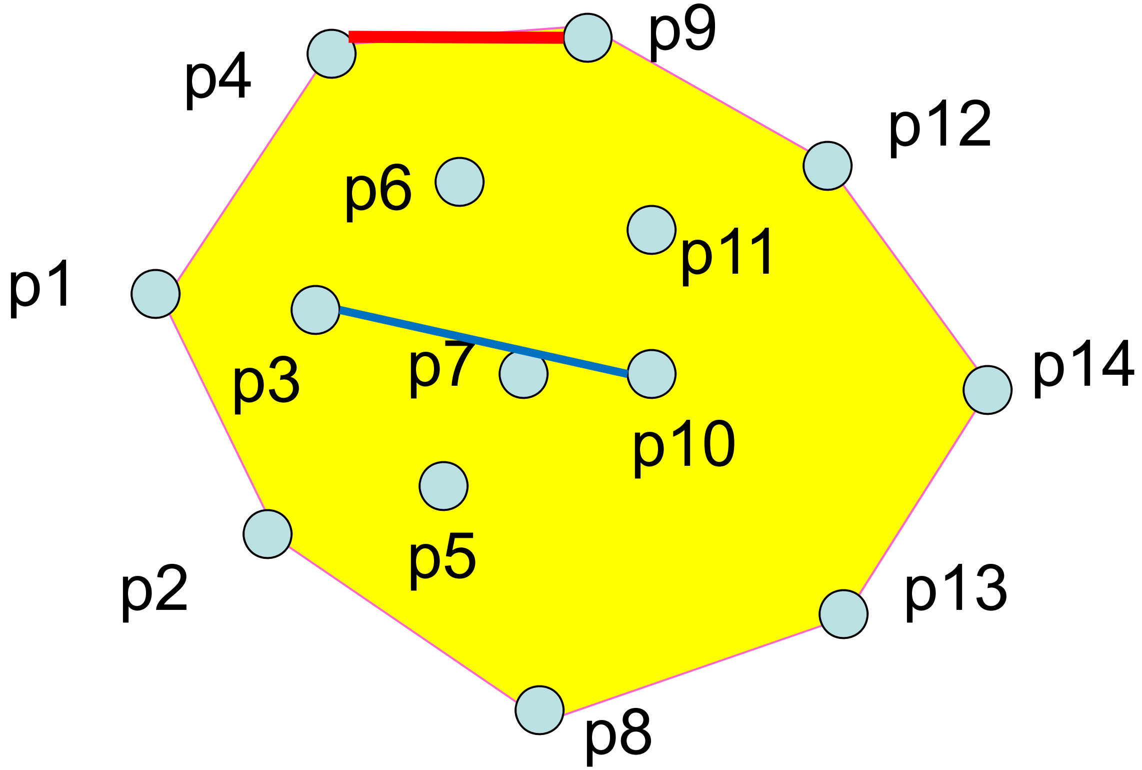

YAMAHA


Edge verification



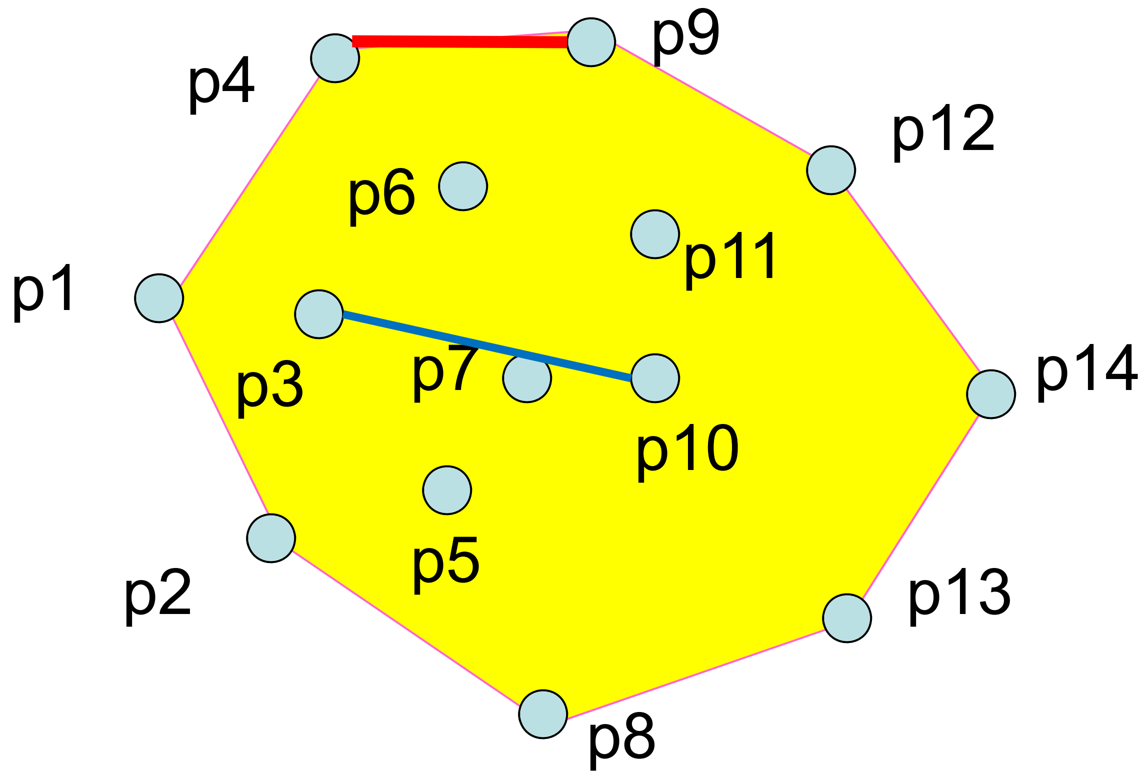
- Order all points in the orthogonal direction to the edge we want to verify
- Convex hull edge if and only if its endpoints are both maximum (or minimum) in the ordering

Edge verification



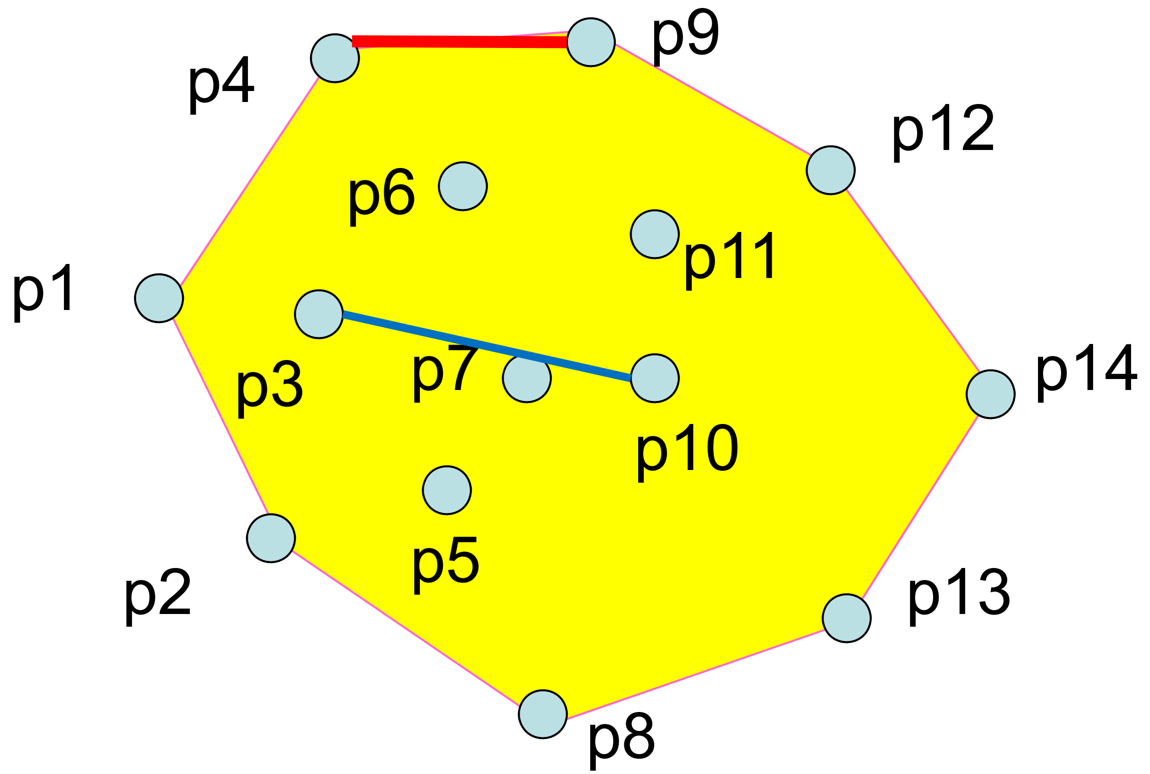
(p4, p9) is an edge of the convex hull, while (p3,p10) is not. How to distinguish them?

Convex hull algorithm using edge verification



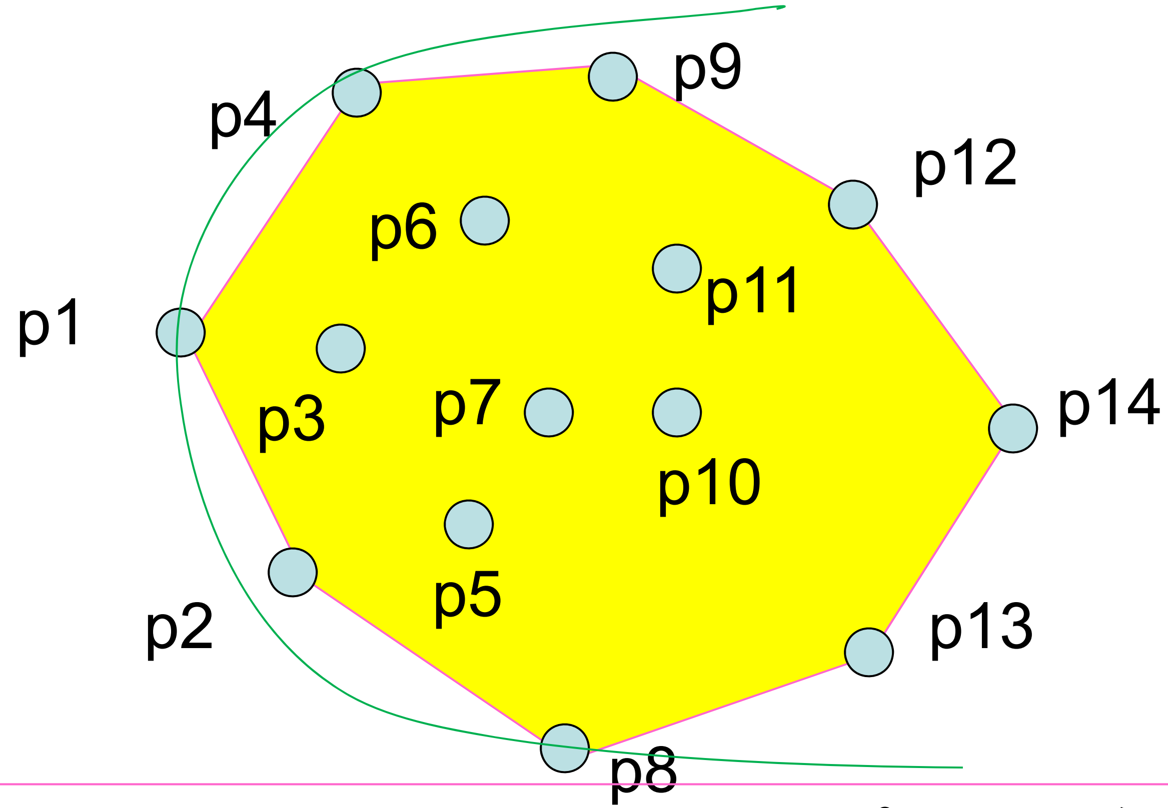
- Verify all n(n-1)/2 candidate edges
- Collect all the convex hull edges
- Arrange them into convex hull
  - How to do it?

Time complexity of the algorithm using edge verification



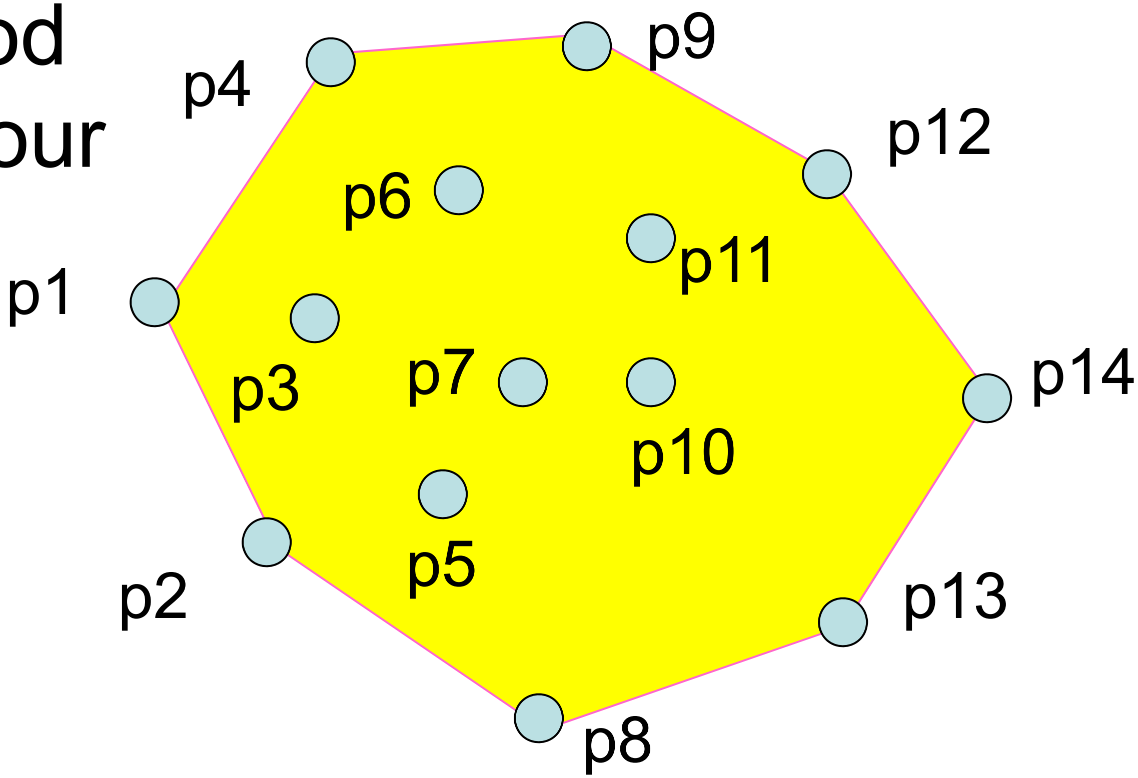
- $O(n^3)$ time algorithm
  - Edge verification = O(n) time for each candidate
  - $O(n^2)$ candidate edges
- Polynomial time. But very slow! How to improve?

Learn from our real life



- Consider points as pins on a board. (パチンコ台の釘だと思おう)
- You are given a string.　How to realize the convex hull. (紐を使って凸包を計算しよう)
- “Gift wrapping” algorithm or………

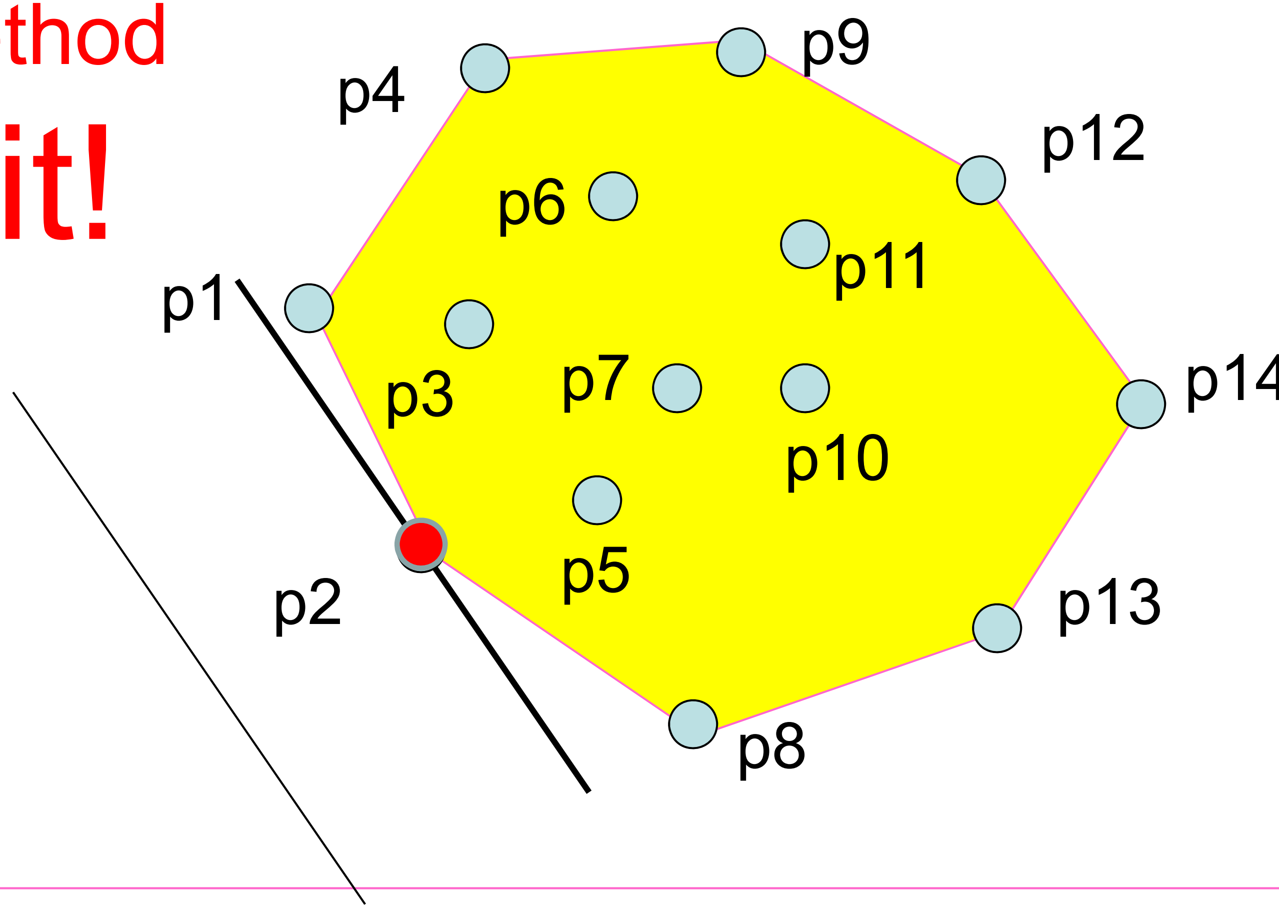
Another method learning from our real life



- Consider points as pins on a board.
- You can touch from any given direction by hand. How to realize the convex hull.
- “hand probing” algorithm

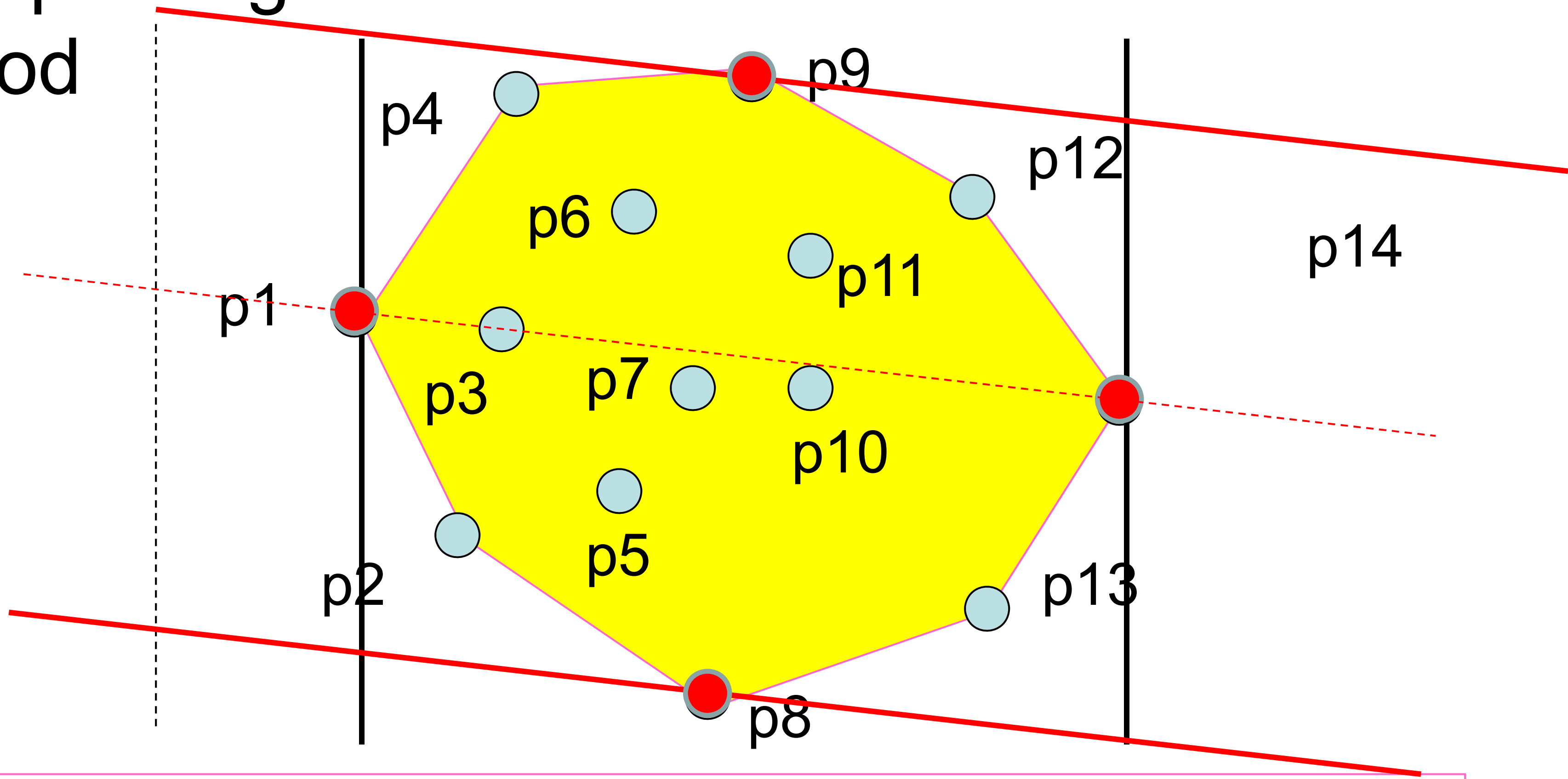
Probing method

I love it!



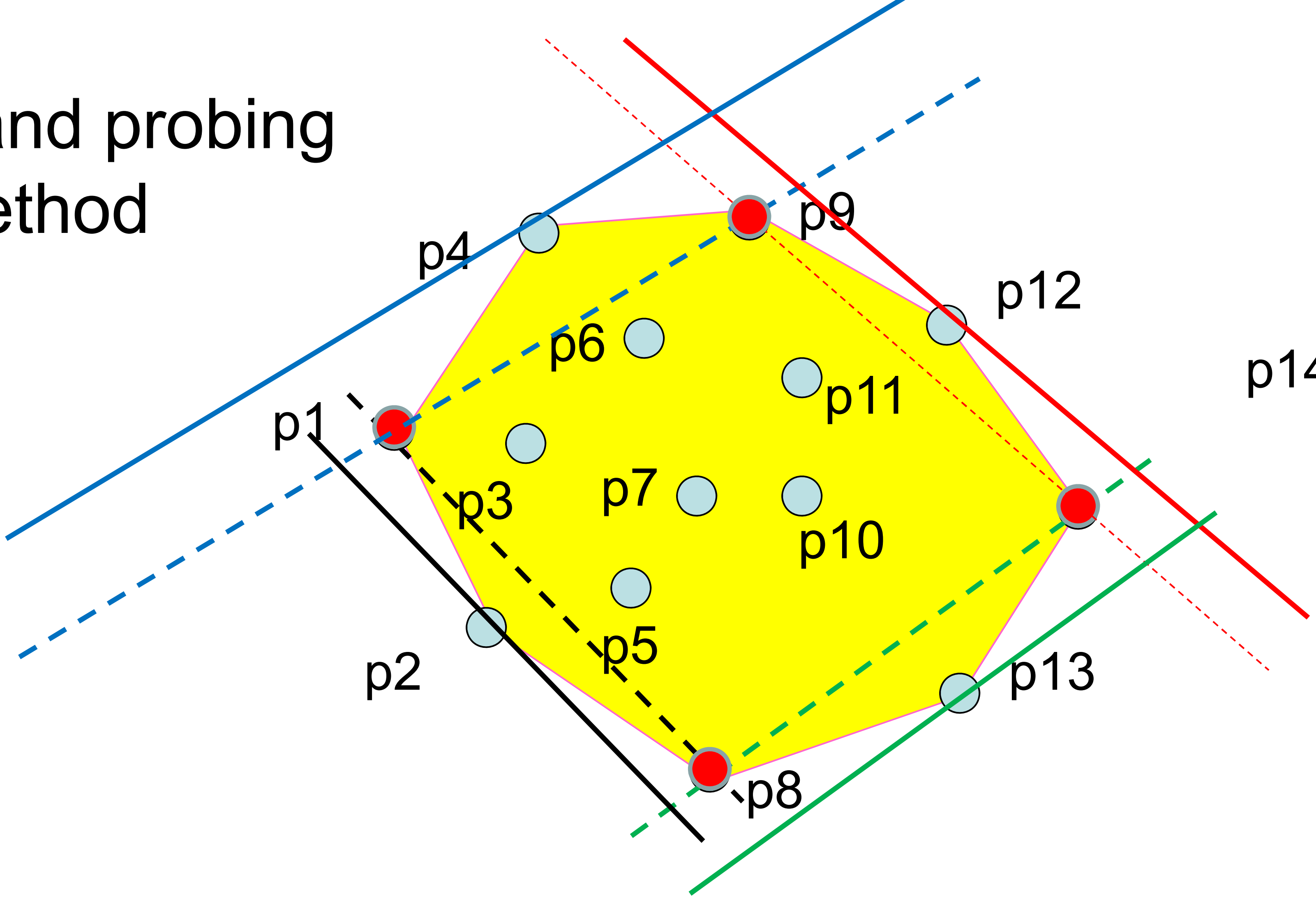
- Consider points as pins on a board.
- You can touch from any given direction by hand to find a vertex of the convex hull.
- "hand probing" algorithm

Hand probing method



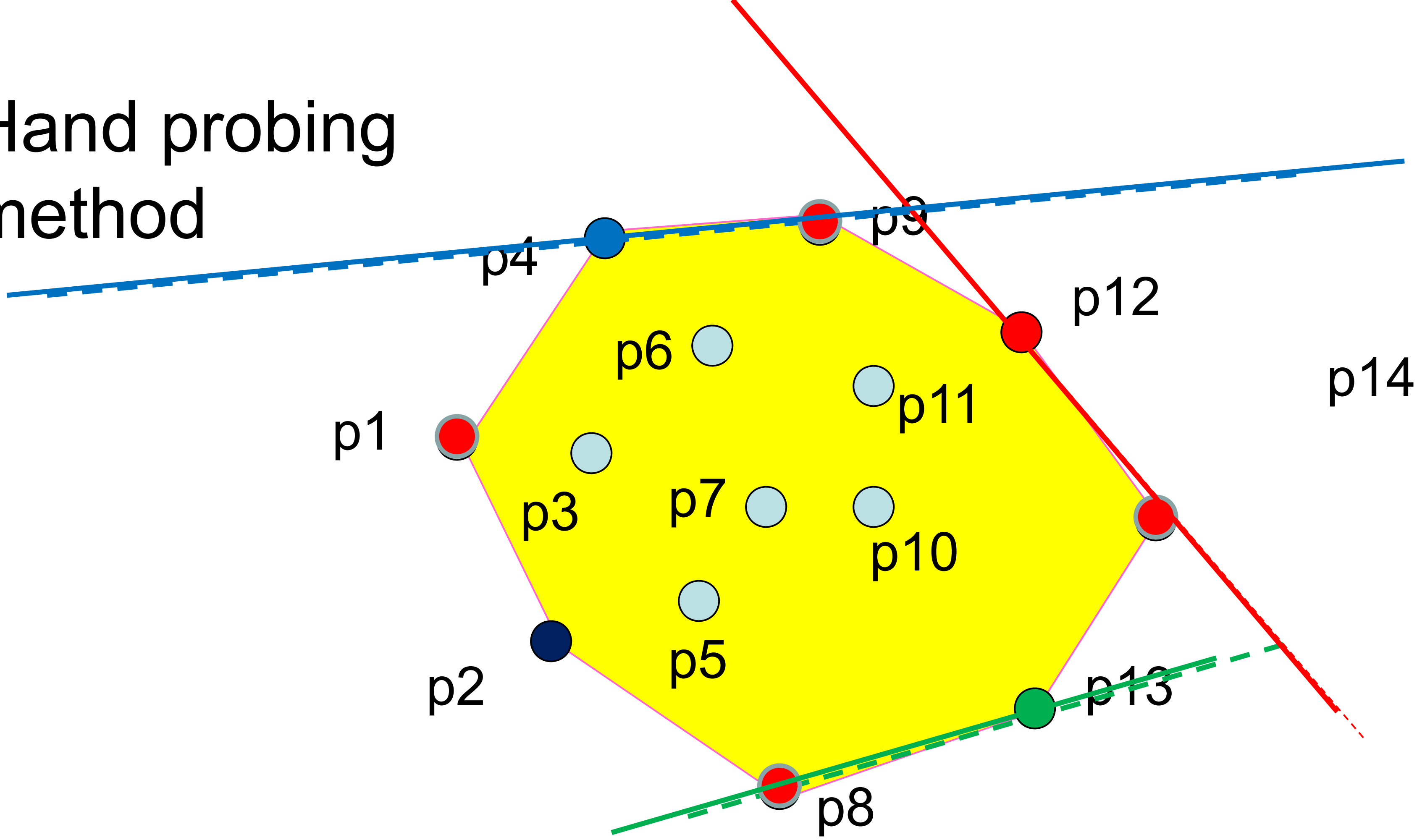
- First touch with vertical lines
- Next, touch with a line parallel to the one between two vertices that we found

Hand probing method



Touch with a line parallel to the one between two adjacent vertices that we found

Hand probing method



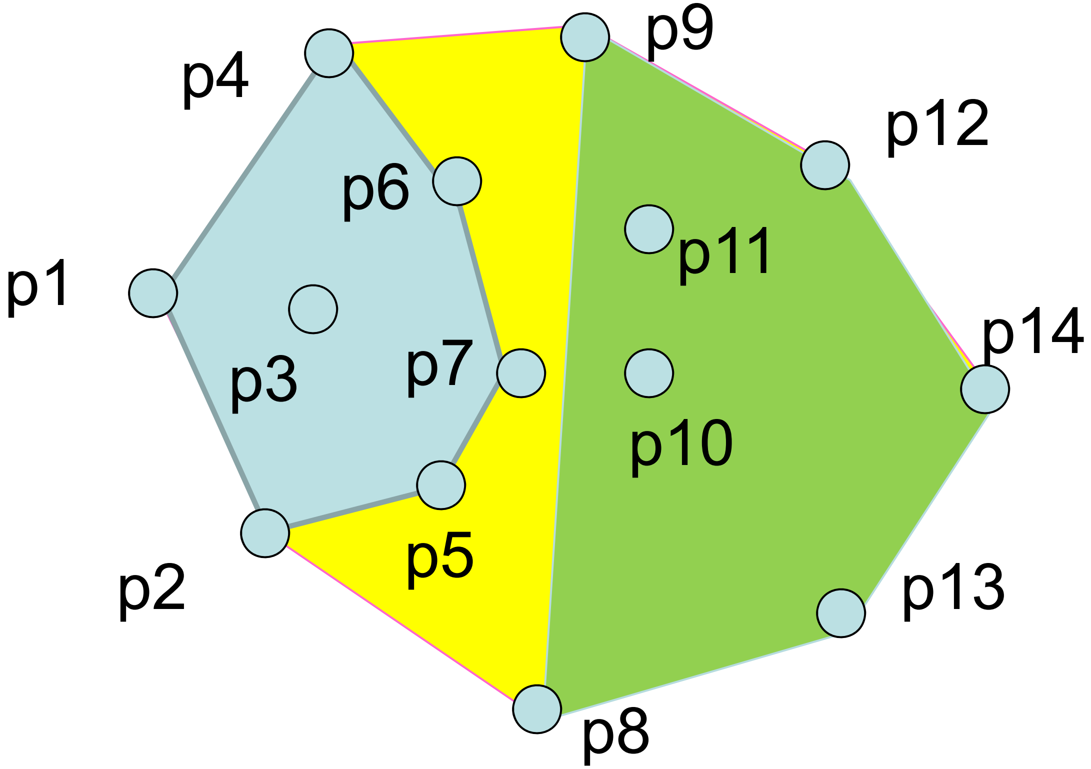
If we do not find a new vertex, we find an edge of the convex hull.

Analysis of probing algorithm

- Each probe can be done in O(n) time
  - Max element finding
- We need at most 2h probes
  - h is the number of vertices in CH(P)
  - We find either a new vertex or a new edge
- Total time complexity O(nh)
  - In the worst case, $O(n^2)$
  - Same as gift wrapping

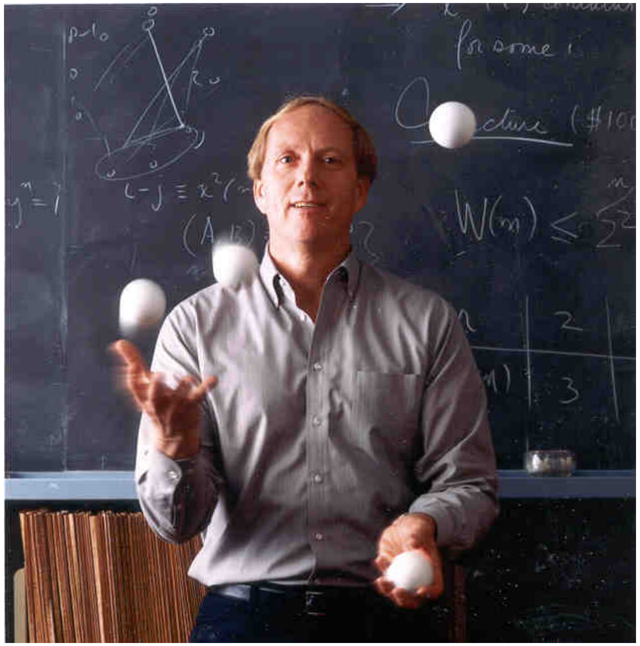
Algorithmic paradigms
1. Divide and conquer

Michael Ian Shamos

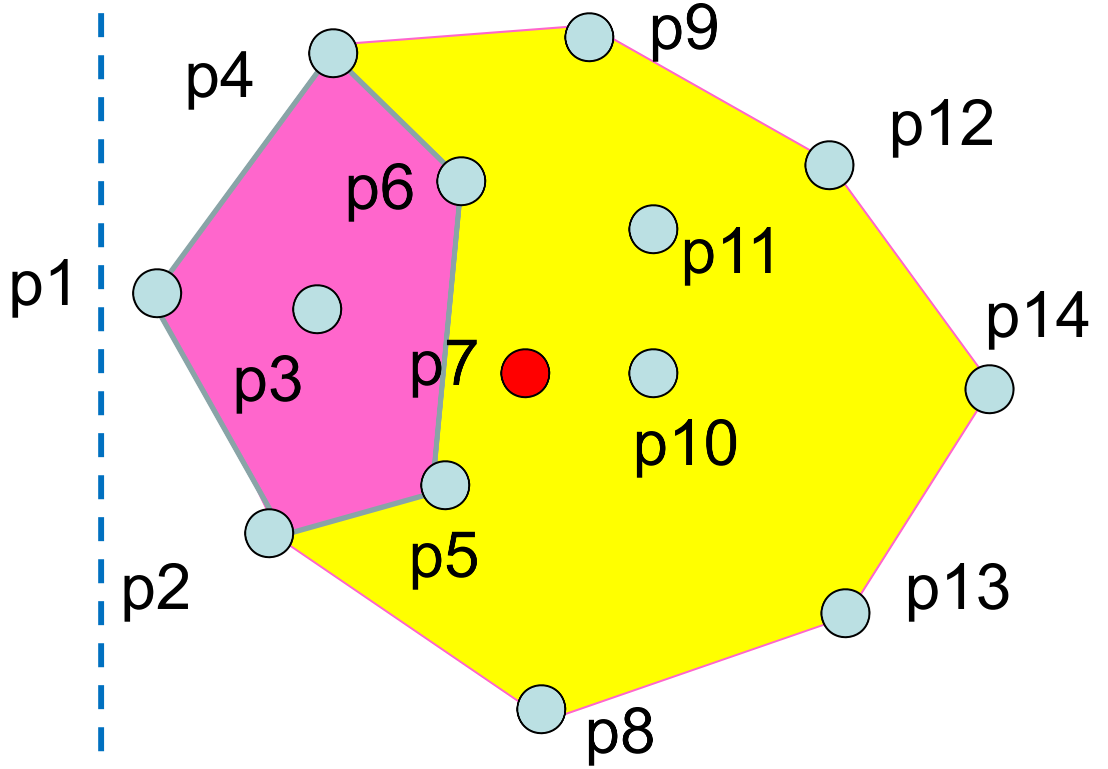


- Process "left half" and "right half" independently
- Merge two outputs

Algorithmic paradigms
2. Incremental method : Graham's scan

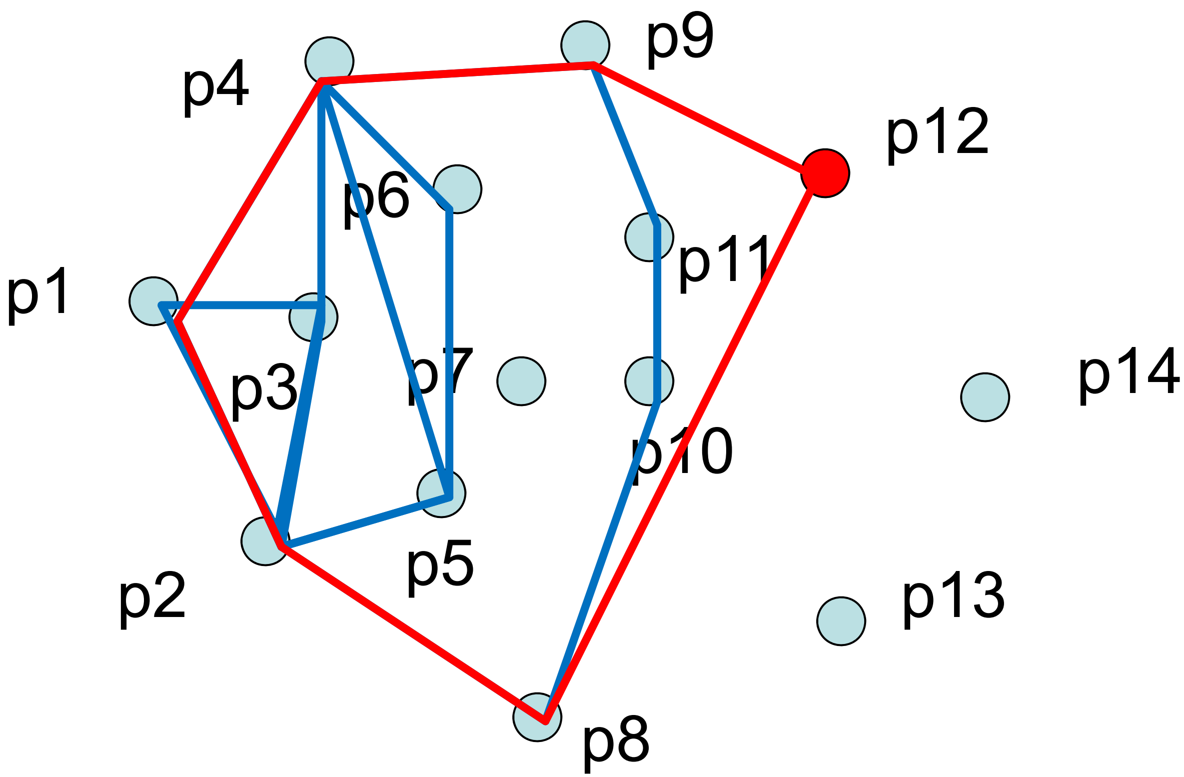


Ronald Graham

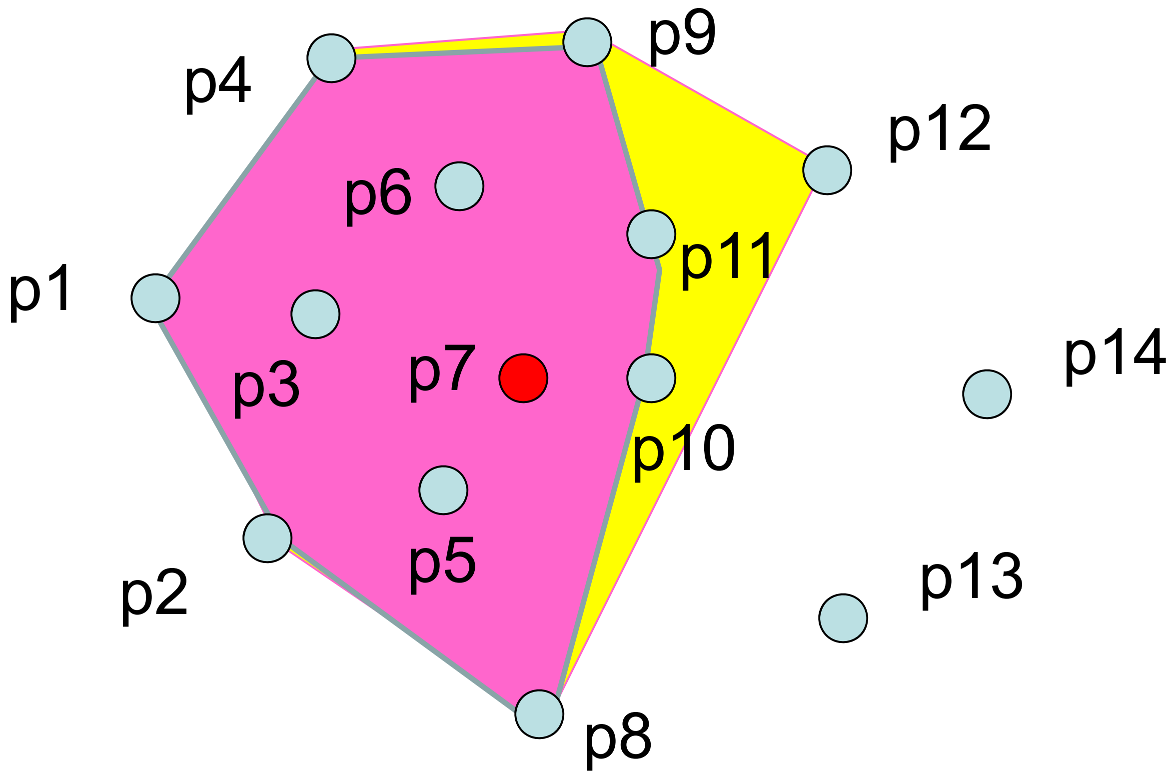


- Process points left-to-right
- Start from the triangle formed by p1,p2,p3, and add points one by one updating the convex hull

Algorithmic paradigms
2. Incremental method : Graham's scan

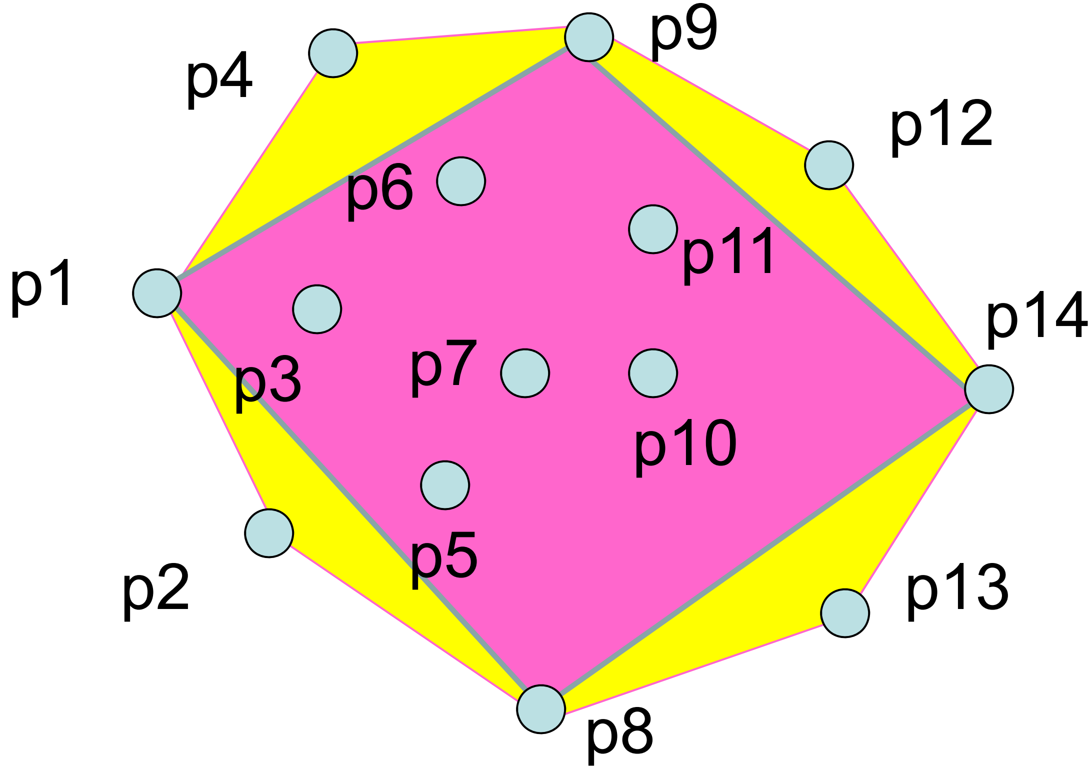


Analysis of the Graham's scan



When we insert a point p(k), We start from p(k-1) to find two tangent points, and remover n(k) vertices between two tangent points It takes O(n(k)) operations. → In total, O(n) operations.

Algorithmic paradigms
3. Prune by preprocessing



- Remove all points in the pink quadrangle
- Run an algorithm discussed before

Time complexities of algorithms

- Brute-force: Exponential time
- Edge verification: $O(n^3)$
- Gift wrapping: $O(n\ h)$
- Probing: $O(n\ h)$
- Divide & Conquer: $O(\ n \log n)$
- Graham’s scan : $O(n \log n)$
- Pruning + Gift wrapping/Graham’s scan: ??

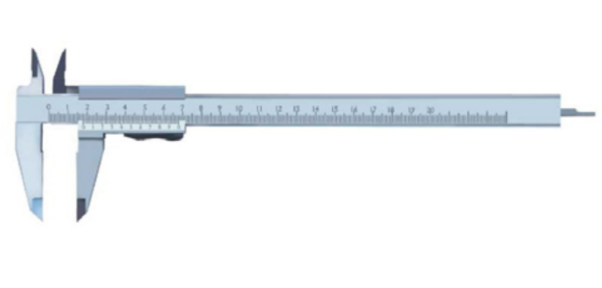
Applications

- Classical use of convex hulls
  – Diameter computation
    - Rotating caliper
  – Fast collision detection

- Tokuyama's original use of convex hull algorithms
  – Statistics
  – Data mining (!)
  – Image processing (!!)

Applications

- Classical use of convex hulls
  - Diameter computation
    - Rotating caliper



Diameter of S = the largest distance of pairs of points.

How to find it? Naively, $O(n^2)$.

We can improve it to O(n log n)

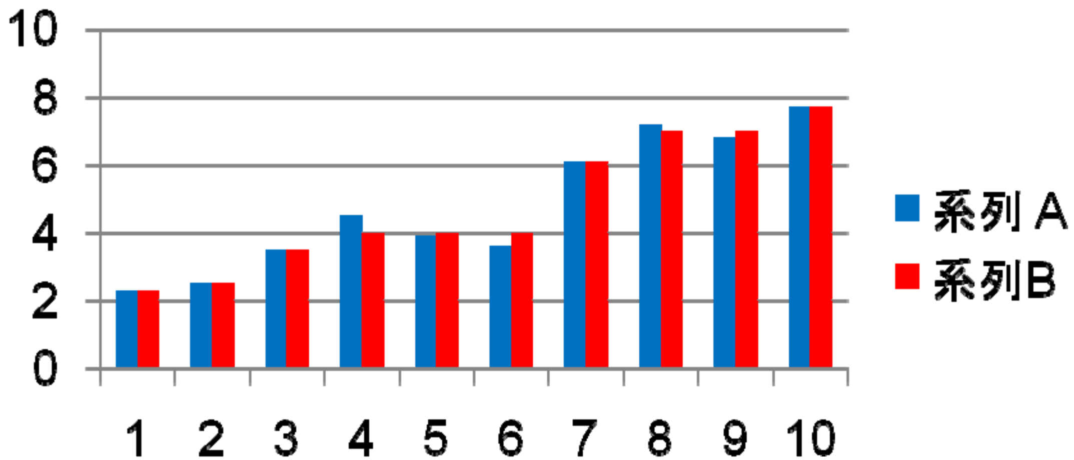
Question to students

- We can compute the largest distance in S in O(n log n) time.
- How about the shortest distance?

- Please think about it…..

Statistics

- Given a histogram A, that is a sequence of real numbers a(1), a(2),….a(n)
- Fing an increasing histogram B: b(1),b(2),..,b(n) approximating A best
  - Minimizing the sum of $(a(i) - b(i))^2$
  - An important operation in statistics
- What is the relation to convex hull?



Statistics

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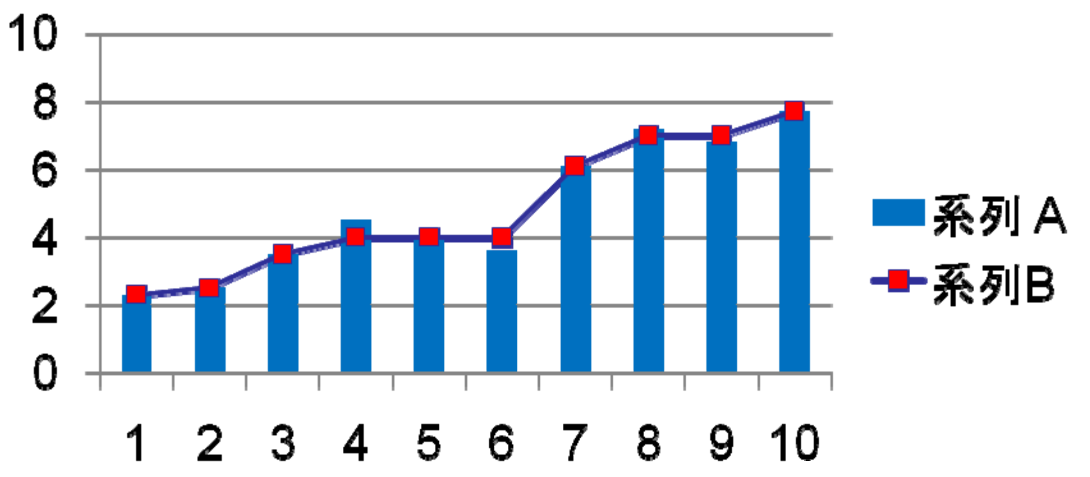


Basic observations

- If an interval A[j,k] = a(j), a(j+1),..a(k) of the histogram is approximated by one value b,b,b,b,..,b, the best value b is the average m(A[j,k]) of the histogram values
- If the optimal increasing sequence approximating A is b,b,b,b,..,b, the average of each prefix a(j), a(j+1),..,a(j+s) cannot be smaller than m(A[j,k])
  - Interval averages are important.

Sequence of prefix sums

- A = a(1), a(2),.., a(n) : input histogram
- sum(i) : a(1)+ a(2)+ ..+ a(i)
- S = { (i, sum(i)): i = 1,2,..,n }
- LCH(S): lower chain of CH(S)
- If you list the slopes of LCH(S) at x = 1.5,2.5,..,n-0.5, we obtainB= (b(i))
- Isn’t it magical?

Sequence of prefix sums

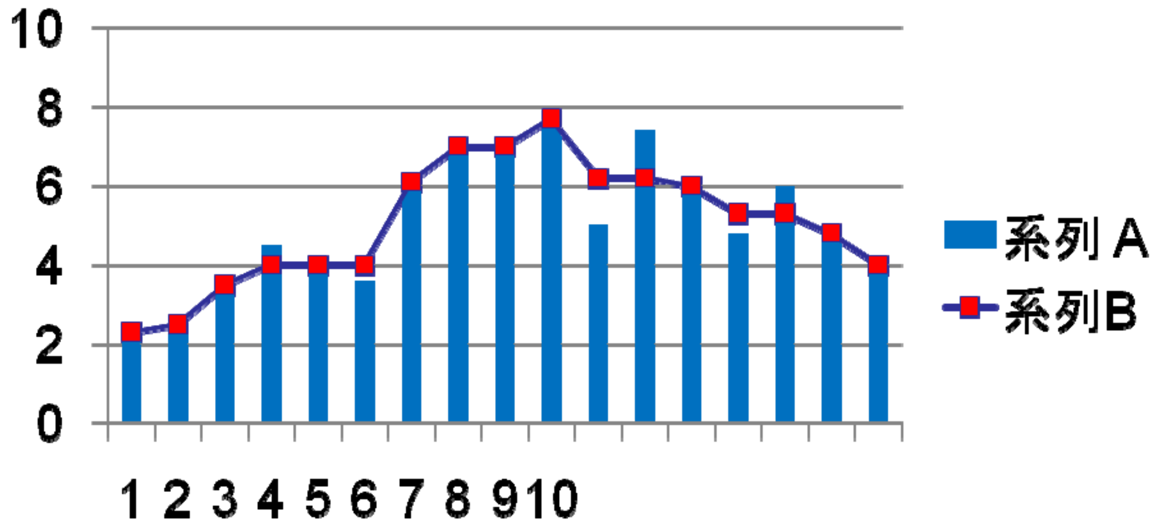
- A = a(1), a(2),.., a(n) : input histogram
- sum(i) : a(1)+ a(2)+ ..+ a(i)
- S = { p(i) = (i, sum(i)): i = 1,2,..,n }
- LCH(S): lower chain of CH(S)
- The slope of p(j) p(k) is the average of a(j+1), a(j+2),..,a(k)
- This leads to the convex hull
  - Explanation: on blackboard.
  - O(n) time algorithm

Approximation of curve

- Input: A piecewise linear curve $y = f(x)$
  - In an interval [0,n], with n linear pieces.
- Output: A nondecreasing curve $y=g(x)$
- Objective: Minimize $|| f - g ||$
- Can we do this? If $f(x)$ is a histogram, we have done it. What is the solution in the general case?
  - Is O(n) time method possible?.

Unimodal approximation

Input: A histogram
Output: A histogram with one maximal peak
Objective: Minimize || f –g ||



Ref 1.: J. Chun, K. Sadakane, T. Tokuyama: Linear Time Algorithm for Approximating a Curve by a Single-Peaked Curve. Algorithmica 44(2): 103-115 (2006)

Ref2: 加藤直樹他 データマイニングとその応用

K-peak approximation

- Input: a piecewise function $y = f(x)$
- Output: a function $y = g(x)$ with at most k maximal peaks
- Objective: Minimize $|| f – g ||$
- This has applications to wave signal processing. (Rhythm finding)